Julia Robinson Mathematics Festival

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Cardinal Infinity

1. What does it mean for two sets to have the same number of elements? Try your definition on the following pairs of sets.

    a. the fingers on one of your hands vs the set {1, 2, 3, 4, 5}

    b. the set of people at this event vs the list of names preregistered

    c. the set {1, 2, 3, 4} and the set of all subsets of {1, 2, 3, 4}
    (The set of all subsets is called the "power set")

    d. the set of all subsets of {a, b, c} and the set of all subsets of {1, 2, 3}

2. Can your method in the previous part deal with infinite sets? Try your definition with the following pairs of sets. Which set is larger, or do they have the same number of elements? How can you tell?

    a. nonnegative numbers vs positive numbers

    b. positive even numbers vs all integers

    c. natural numbers vs pairs of natural numbers

    d. natural numbers vs rational numbers

3. All the sets in above have the same "cardinality", i.e. they are "equally infinite". All sets with cardinality equal to some subset of the natural numbers (including finite ones) are said to be "countable". Note: A set is countable iff one can write out ALL its members in a sequence (finite or infinite).

    a. Show that the set of real numbers is NOT countable

    b. Show that the set of all subsets of the natural numbers is NOT countable

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4. Definition: A collection F of subsets of natural numbers is said to be GOOD, if the intersection of any two members of F is finite. We are interested in finding the largest possible GOOD collection F.

   a. Construct a finite collection F that is GOOD.

   b. Construct a finite collection F that is **not** GOOD.

   c. Prove that if F is good then F is at most the same cardinality as the power set of the natural numbers.

   d. There exists a good F, such that Card(F) is infinite (e.g. come up with a countable F).

   e. Is there a good F, such that Card(F) is uncountable? Hint: the real numbers are uncountable.

Advanced Problems

5. What is the largest collection of subsets of a set of 100 elements such that any two subsets have:

   a. no intersection? (easy)

   b. at most one element of intersection?

   c. at most two elements of intersection?

6. Find a collection of subsets where all pairwise intersections have even cardinality.