Julia Robinson Mathematics Festival

Directed By Joshua Zucker

Counting: Deranged?

An arrangement is a way of organizing things: a derangement is a way of disorganizing things, generally. If you have n things—say, hats—and n people, each of whom owns one of the hats, let's call $D(n,k)$ = the number of ways of distributing the hats, one to each person, so that exactly k people get the correct hat and the other $n-k$ get the wrong hat.

1. Compute $D(n, n)$.

2. Compute $D(n, n-1)$. Yes, that's a little math humor there.

3. Compute $D(1, k)$ for all relevant k. Which k are relevant?

4. Compute $D(2, k)$ for all relevant k. Start organizing your work in a table.

5. Compute $D(3, k)$ and $D(4, k)$ for all relevant k. Which k are relevant now? At least up to here, even if you have no fancy ideas, you should be able to answer these questions by listing every possibility. But how to continue?

6. What is the sum of each row of the table? That is, what is the total of D(4, something) for example? Can you explain why it should be this way?

7. Look for and explain a relationship between $D(n, k)$ and D(smaller values of n). Knowledge of Pascal's triangle should be very helpful. Perhaps your method fails for $D(n, 0)$; if so, that's OK.

8. Now figure out a way to compute $D(n, 0)$ and fill out your entire table.

9. For an alternate method, let A_j be the set of all the hat arrangements in which person j (and perhaps others as well) gets the right hat. Then visit the "Counting: as easy as PIE" table to find the number of hat arrangements total among all the A_j and therefore to find $D(n, 0)$ as the number of hat arrangements that aren't in any of the A_j.

10. Can you explain what this has to do with the number e?

11. On average, how many people get the correct hat? Compute it for small values of n and describe a pattern (at least one that is accurate for large values of n).