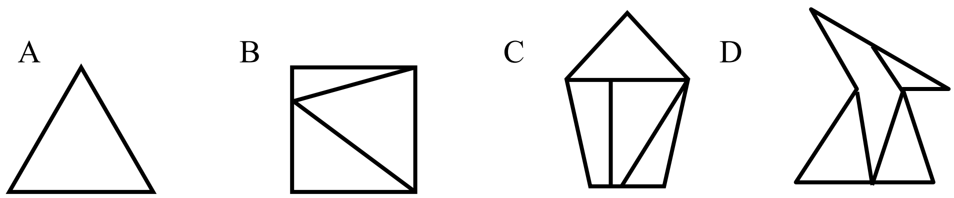
# Dissections

A **dissection** of a polygon is a decomposition of the polygon into finitely many polygons (called **pieces**). In Figure 1, the triangle A and quadrilateral B are dissected into triangles. The pentagon and the hexagon are each dissected into four pieces.

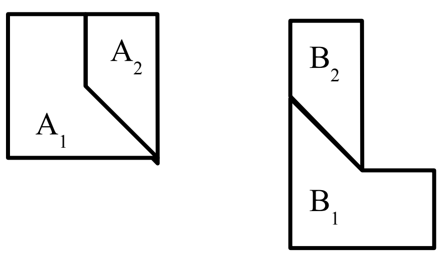
Figure 1




1. Draw some quadrilaterals, pentagons, hexagons, heptagons and octagons and dissect them into triangles.
2. Can any polygon with N sides be dissected into triangles for all values of N? Be sure your answer works for polygons that are not convex (a convex polygon is one in which all interior vertex angles are less than 180°). If so, what is the minimum number of triangles necessary? If not, give an example of a polygon that cannot be dissected into triangles.

Two polygons A and B are **congruent by dissection** if A can be dissected into pieces $A_1$, $A_2$, $A_3$,…, $A_n$ and B can be dissected into pieces $B_1$, $B_2$, $B_3$,…, $B_n$ such that $A_1$ ¿ $B_1$, $A_2$ ¿ $B_2$, …, $A_n$ ¿ $B_n$ (where ¿ means congruent to).

Figure 2

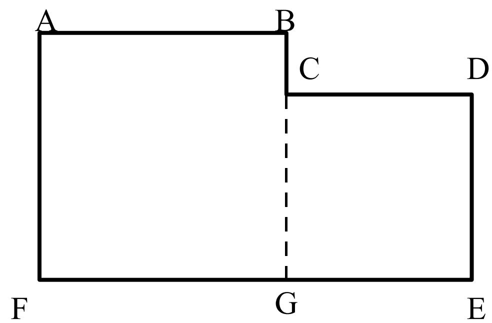



The square and the L-shaped hexagon in Figure 2 are congruent by dissection. Two polygons that are congruent by dissection have the same area.

3. Suppose right triangle ABC ($m\angle B=90°$) and rectangle DEFG have the same area and that AB=DE. Show that they are congruent by dissection.
4. Suppose obtuse triangle ABC ($m\angle B>90°$) and rectangle DEFG have the same area and that AB=DE. Show that they are congruent by dissection.
5. Suppose acute triangle ABC and rectangle DEFG have the same area and that AB=DE. Show that they are congruent by dissection.
6. Suppose rectangle ABCD has side lengths AB=CD=12 and BC=AD=3. Show that ABCD is congruent by dissection to a square whose side is 6.

7. Suppose rectangle ABCD has side lengths AB=CD=9 and BC=AD=4. Show that ABCD is congruent by dissection to a square whose side is 6.
8. Suppose rectangle ABCD has side lengths AB=CD=25 and BC=AD=4. Show that ABCD is congruent by dissection to a square whose side is 10.
9. Show that *any* rectangle is congruent by dissection to a square of the same area.
10. In Figure 3 below, the hexagon ABCDEF is comprised of two adjacent squares (ABGF and CDEG). Show that ABCDEF is congruent by dissection to a square. (Hint: Pythagoras might have used this dissection to prove his famous theorem.)

Figure 3



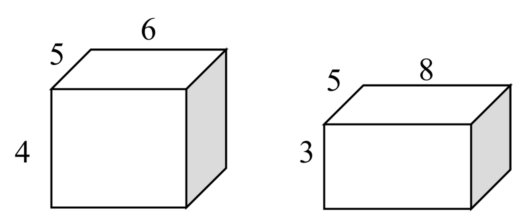

11. Use your results from problems 2, 3, 7, and 8 to show that *any* two polygons with the same area are congruent by dissection!

The result from problem 9 is known as the Bolyai-Gerwien Theorem and was first proved in the 1800's.

## Three-Dimensional Dissection

12. Three-dimensional dissection of a polyhedron is defined analogously to polygon dissection (each piece of the dissection must be a polyhedron). Show that a 4x5x6 rectangular prism is congruent by dissection to a 3x5x8 rectangular prism.

Figure 4




13. Show that a 3x25x45 rectangular prism is congruent by dissection to a 15x15x15 cube.
14. Show that a 24x25x45 rectangular prism is congruent by dissection to a 30x30x30 cube.
15. Show that *any* rectangular prism is congruent by dissection to a cube of the same volume.
16. Max Dehn proved in 1900 that, unlike for polygons in two dimensions, a regular tetrahedron is *not* congruent by dissection to a cube of the same volume, solving the third of Hilbert's famous problems.