

Making Infinity Finite

Our first set of problems has to do with taking what appears to be an infinite process and finding a way to "short-circuit" it so that it is actually finite. Choose just a few of these that look like the most fun; they go roughly in increasing order of difficulty.

1. Compute the value of $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$

2. Compute the value of $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}$

3. Compute the value of $2 + \frac{3}{4 + \frac{3}{4 + \frac{3}{4 + \dots}}}$

4. Compute the value of $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$. You may know some more conventional ways to solve this; in what ways is it like the problems above?

5. Compute the value of $\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots$. Can you use the answer to the previous problem to help?

6. Compute the value of $1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\dots}}}}$

7. Compute the value of $\sqrt{3\sqrt{9\sqrt{27\sqrt{81\sqrt{\dots}}}}}$

8. Compute the value of $\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2007 - \dots}}}$

Now let's apply what we've learned about infinity to solve some problems about flipping coins.

9. On average, how long do you have to wait when flipping a coin before you get the first head? Hint: Let x stand for the average number of flips you'll have to wait.
10. What is the probability of getting HH in two flips? HT?
11. How many times do you have to flip a coin on average to get HH?
12. How many times do you have to flip a coin on average to get HT?
13. If you bet on HT happening first, and I bet on HH happening first, what are your odds of winning?
14. What is the probability of getting HHH in three flips? HTH? HHT?
15. How many times do you have to flip a coin on average to get HHH?
16. How many times do you have to flip a coin on average to get HTH?
17. How many times do you have to flip a coin on average to get HHT?
18. If you bet on HTH happening first and I bet on HHH, what are your odds of winning?
19. What about HHH vs HHT?
20. And HTH vs HHT?

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