Prisoner Probabilities, Certainties, and a Paradox

Probabilities

1. Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns, in any way he liked, after which he had to draw a ball at random from one of these urns. The story goes that he drew a white ball and was released. How did he maximize his chances of drawing a white ball, and what was his chance of success?

2. Ten convicted murderers are to be shot at dawn. However, to give them a chance of survival, the 10 members of the firing squad each randomly pick someone to shoot at, with a 1/10 probability of selecting each murderer. Each member of the firing squad shoots only one bullet which is always lethal. Anyone murderer who is not shot in this process gets life in prison instead of execution by firing squad.

   a. What is the probability that all 10 members of the firing squad shoot the same murderer?

   b. What is the probability that all 10 members of the firing squad shoot different murderers?

   c. What is the probability that exactly one murderer survives?

   d. On average, how many of the murderers survive to "enjoy" the rest of their life in prison?

Certainties

3. A group of prisoners sits in a circle, numbered from 1 to n. Going around the circle repeatedly, the warden alternately says "skip, cell, skip, cell," and so on, until finally only one person remains. That person is released from the prison. For instance, with 4 prisoners, you skip prisoner 1, send prisoner 2 to the cell, skip prisoner 3, send prisoner 4 to the cell, skip prisoner 1, send prisoner 3 to the cell, and thus prisoner 1 is released.

   a. In fact, on this day prisoner 1 is released, but the total number of prisoners is not 4: it is between 40 and 100. How many prisoners are there?

   b. On another day, when again there are between 40 and 100 prisoners, the very last prisoner (number n) is released. What is the value of n? Is there only one possibility?

   c. On yet another day, prisoner 7 is released. How many prisoners were there that day? Make sure to analyze all possibilities.

d. Generalize: find a rule for which prisoner number you want to be, if you know how many prisoners will be sitting in the circle.

e. What if the rule is changed, and instead of the last prisoner being released, it is the second-to-last prisoner who is released? Now what should be your strategy?

4. A new warden is hired whose favorite number is 7. Thus, instead of counting "skip, cell, skip, cell", the new warden counts "1, 2, 3, 4, 5, 6, cell," over and over again, until all but one prisoner has been sent back to the cells. The one remaining prisoner is released. For example, with four people, the warden counts 1, 2, 3, 4, on person number 1, 2, 3, 4. Then the warden continues counting 5, 6, cell, so person 3 is sent to the cell. Continuing, person 4 is sent to the cell next, and then person 1, so number 2 is the desired number.

   a. One day there are 25 prisoners. What seat is the desired one?

   b. If you know how many prisoners there are, find a quick way to decide which number will go free.

Paradox

5. Every Sunday, an extremely trustworthy jailer announces to a prisoner "At noon one day this week, Monday through Friday, you will be selected for a surprise release." For many weeks, indeed, the jailer has made this announcement and the prisoner to whom the announcement was made was indeed surprised to be selected at noon one day and then executed. But this week, the jailer makes the announcement to a mathematician.

   a. The mathematician reasons that if the release does not happen on Monday, Tuesday, Wednesday, or Thursday, then when Friday arrives the release announcement will be expected, not a surprise to anyone. Hence, the announcement can never come on Friday. Is this reasoning correct?

   b. Now that Friday is guaranteed not to be the day of the release, the mathematician reasons that it can't happen on Thursday, either. For if Monday, Tuesday, and Wednesday have gone by without a noon announcement, and if Friday is impossible, then on Thursday the announcement will not be a surprise.

   c. The mathematician continues reasoning in the same way and determines that Wednesday is also impossible. Explain the mathematician's reasoning.

   d. Continuing that reasoning, Tuesday and Monday are also impossible. Explain.

   e. Yet, at noon on Wednesday, the mathematician is called for release and, based on the above reasoning, is certainly surprised after all! What is wrong with the mathematician's reasoning?