Proof by Zomes

Mathpath 2007
Dr. Z (zeitz@usfca.edu)

The problems below scratch the surface of three wonderfully connected things: the golden ratio, the geometry of the platonic solids, and zometool construction set. Our goal is to discover, mostly by playing, how to compute the volume of a regular dodecahedron.

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1 *Get familiar with zomes*. Zometools consist of balls and struts. The struts come in (essentially) three colors: blue, yellow, red. Each color of strut comes in (essentially) three lengths: short, medium, long. We use subscripts to indicate size. Thus b_1 is a short blue, and r_3 is long red.

Your first assignment: locate all nine kinds of primary-color struts, and then build

(a) an equilateral triangle with blue sides.

(b) a b_1, b_1, b_2 triangle.

(c) a b_2, b_2, b_1 triangle.

(d) a y_1, y_1, b_1 triangle.

(e) a r_1, r_1, b_1 triangle.

(f) a square.

(g) a regular pentagon.

(h) a regular hexagon.

2 *Length computation*. The length of a strut is not its physical length. Instead, we put balls on each end and define its length to be the distance between the centers of these two balls. This is not a weird definition; it is the only sensible definition, since zome objects are built with struts AND balls.

(a) Using this definition, verify that

$$x_1 + x_2 = x_3$$

for any color b, r, y.

(b) Try making “medium” and “large” versions of some of the triangles that you built earlier. Use similar triangles to conclude that

$$\frac{b_3}{b_2} = \frac{b_2}{b_1}$$

and

$$\frac{y_2}{y_1} = \frac{r_2}{r_1} = \frac{b_2}{b_1}$$

and

$$\frac{y_3}{y_2} = \frac{r_3}{r_2} = \frac{b_3}{b_2}.$$

Call this common ratio τ.

(c) Prove that

$$\tau^2 = \tau + 1,$$

and hence τ is the famous *golden ratio*

$$\tau = \frac{1+\sqrt{5}}{2}.$$

(d) For an alternative derivation, build a large version of the isosceles triangle of 1(c), i.e., use lengths of b_3, b_3, b_2. However, replace one of the b_3 sides with a b_2 and a b_1, and then use another b_2 to make the famous similar triangles inside the *golden triangle*.

3 *Digression on golden ratio algebra*. The formula $\tau^2 = \tau + 1$ means that any polynomial in τ can be reduced to a linear expression in τ. Practice with this; show that $\tau^3 = 2\tau + 1$ and $1/\tau = \tau - 1$. Find a simple expression for τ^7. See anything familiar?

4 *Fun with cubes*. Build a cube with blue struts.

(a) Then use yellow struts to make the long diagonal. Use this construction to show that

$$y_i = \frac{b_i\sqrt{3}}{2}.$$

(b) Recall that the formula for the volume of a pyramid is

$$V = \frac{Bh}{3},$$

where B is the area of the base and h is the height. Why is this formula true?

5 *Duality*.

(a) Build a dodecahedron using b_1 struts. A dodecahedron consists of regular pentagons that meet three to a vertex. Carefully count the number of faces, edges, and vertices.

(b) Now build an icosahedron, which consists of equilateral triangles that meet five to a vertex. Carefully count the number of faces, edges, and vertices.

(c) For each of the above polyhedra, locate the center and hold a ball there. Notice that you can join each vertex to this central ball with struts of the same color. Do it.

(d) Now, what if you took the central ball of one of your polyhedra, and put in the "other" color of strut, but at a bigger size? What do you get? Build it, and ponder.

6 *Hidden in polyhedra*. Most people don't know about the following wonderful phenomena. Discover for yourself the

(a) tetrahedra hidden naturally in the cube (you will need green struts for this)

(b) golden rectangles in the icosahedron

(c) cubes in the dodecahedron

7 *The volume of the dodecahedron*. Let us find a formula for the volume of the dodecahedron with side length 1.

(a) Once you have discovered the cubes that lurk in the dodecahedron, you can think of a dodecahedron as a cube plus 6 identical "roof structures." Verify that the side length of the cube is τ. Thus, if we can compute the volume R of a roof structure, the volume of the entire dodecahedron will be $\tau^3 + 6R$.

(b) Use the Pythagorean theorem to show that the height of the roof structure is $1/2$. Problem 3 may save you some pain.

(c) The roof structure can be broken into three parts: a central triangular prism, with identical structures on either side of the prism which can be shoved together to form a pyramid with a rectangular base.

(d) Show that the volume of the prism is $\tau/4$.

(e) Show that the volume of this pyramid equals 1/6, and that the base is not just any rectangle, but—you guessed it—a golden rectangle.

(f) Put it all together to show that the volume of the dodecahedron is

$$\frac{8+7\tau}{2} = \frac{15+7\sqrt{5}}{4}.$$

8 *A problem from the Bay Area Mathematical Olympiad*. The following problem was posed by Gregory Galperin for BAMO 2005. We were not sure if anyone could solve it, but we wanted to use the problem because it made a great T-shirt.

> Let D be a dodecahedron which can be inscribed in a sphere with radius R. Let I be an icosahedron which can also be inscribed in a sphere of radius R. Which has the greater volume, and why?

Indeed, no one found a solution. Can you, using duality?