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Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. Line segments OA and OB meet at O with an angle of 1 degree. Points C and D bisect OA and OB respectively. Imagine that the segments CA and DB act as reflective mirrors, and that a light ray in the plane enters the larger opening AB, bounces back and forth between the two mirrors for a while and then exits. What is the maximum number of reflections that are possible?

Discussion: Draw 360 radii of a circle centered at O, with the angle between each of the radii and its neighbors being one degree. Draw another, smaller concentric circle whose radius is half of the bigger circle. Then call the outer half of each of the 360 radii a "spoke." Partition the spokes alternately into even spokes and odd spokes.

Now imagine that the mirrors are only semi-reflective, so that when a light ray strikes, it divides into a reflected component (which bounces off the mirror) and a transmitted component, which continues on through the mirror as a straight line continuation of the incoming ray. Since the angle of reflection equals the angle of incidence, the straight-line continuation and the reflected components will both strike their respective next neighboring mirrors at exactly the same distance from center O. Continuing this same line of argument, we soon find a one-to-one correspondence between the light ray which might be reflected back and forth between CA and DB and the straight line.

So the question of how many times a ray might be reflected back and forth between CA and DB translates into the question of how many different spokes might be intersected by a single line which never goes inside the inner circle. The line which is tangent to the inner circle is clearly one strong candidate. The number of degrees subtended by this line is

$$2 \cdot \arccos(1/2) = 120°.$$

So evidently, a light ray can reflect off of AC and BD a total of 120 times, and no more. Each of the two mirrors accounts for 60 of these reflections.

121 reflections is marginally conceivable, if one reflection precisely at point C and two reflections precisely at point A are all allowed. But since any real light wave has a finite (positive) wavelength, we think that this conceptual possibility deserves to be rejected.

Problem 2. (Just when you thought you were safe from hat problems...) Let n be a positive integer. A team of n people has a strategy session. After that a judge randomly places red and blue hats on their heads. Each person can see all hats except their own (and sees who is wearing which hat, in addition to the color of the hat). *Simultaneously, and with no communication whatsoever*, each player must vote on their hat color (abstention is not allowed). However, they follow "Chicago" rules: each player can place as many votes as they desire. If a majority of the players are correct, the team shares one million dollars.

Find a strategy that maximizes the team's chances of winning the million dollars.

Discussion: First, we correct an error in the statement of the problem, whose second-to-last sentence should have read, "If a majority of the *votes* are correct, the team shares one million dollars." Fortunately, most readers understood the context and interpreted the problem as had been intended.

This problem apparently goes back at least to J. Aspens, R. Beigel, M. Furst and S. Rudich, "The expressive power of voting polynomials", *Combinatorica* **14**:2 (1994), 1–14.

Here is a solution that wins with probability $1 - 2^{-n}$.

In their strategy session, the team assigns index numbers to its members, ranging from 0 to $n-1$. The team will lose only when all of the hats are red; in that case there will be a very large number of votes cast, and all of them will be wrong. When there is at least one blue hat, then the smallest-numbered player with a blue hat guesses correctly, and he casts a sufficiently large number of votes to dominate the election. Each of the players with a larger index number casts as few votes as the election rules permit. More specifically, if the k-th player sees any blue hat(s) preceding him (i.e., on players with smaller indices) he casts the minimum number of votes permissible, which we denote by m. But if he sees no blue hats preceding him, then he casts $f(k)$ votes in favor of his own hat being blue, and $f(k)$ will be large enough that his votes alone are sufficient to dominate the election if he is lucky enough to be correct.

This strategy is most easily exemplified in a simpler version of the problem, in which abstention *is* permitted, so $m = 0$. In this case, we can choose $f(k) = 2^k$.

More generally, our strategy works only if

$$f(k) > F(k) + m(n-k-1)$$

where

$$F(k) = \sum_{j=0}^{k-1} f(j).$$

That's because $F(k)$ is the number of incorrect votes casts by preceding players, and $m(n-k)$ is the maximum possible number of incorrect votes cast by subsequent players. The minimum function $f(k)$ evidently satisfies this recursion:

$$f(k) = 1 + F(k) + m(n-k-1).$$

When m = 1, this becomes $f(k) = n - k + F(k)$ whence

$$\begin{aligned} f(0) = n &= n \\ f(1) = n + n - 1 &= 2n - 1 \\ f(2) = 3n - 1 + n - 2 &= 4n - 3 \\ f(3) = 7n - 4 + n - 3 &= 8n - 7 \\ &\cdots \\ f(k) &= 2^k(n-1) + 1 \end{aligned}$$

as is easily proved by induction.

Problem 3. Show that for any positive integer n the regular $2n$-gon of side 1 can be tiled by rhombuses of side 1. How many rhombuses did you use?

Problem 4. You are a playwright and want to write a play, for a company of n actors, in which actors will enter and exit one at a time, subject to the following constraint: any actor that exits is the one that has been on stage the longest.

For $n = 3$, $n = 4$, and $n = 5$, can you devise a sequence of entrances and exits in which each set of actors appears once and only once? The play begins with an empty stage, so the initial subset is the empty set. You get extra credit if the final set contains a single actor, because then the sequence could cycle, like a Beckett play.