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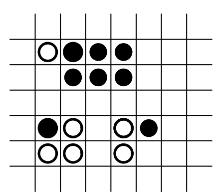
Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. A 12 inch length of wire is to be cut into pieces that are used to construct the frame (i.e., the edges) of a cube with 1-inch sides. What is the smallest possible number of pieces into which the wire must be cut?

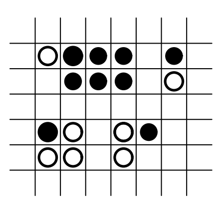
Discussion: Since 3 edges lead into each of the 8 vertices, there are at least 8 "ends," and therefore there are at least 4 segments, i.e., 4 wires. It is easy to construct 4 segments that achieve this minimum.

Problem 2. The new combinatorial game "Clobber" is played by two players, white and black, on a rectangular *m* by *n* checkerboard. In the initial position, all squares are occupied by a stone, with white stones on the white squares and black stones on the black squares. A player moves by picking up one of their stones and clobbering an opponent's stone on an adjacent square (horizontally or vertically). The clobbered stone is removed from the board and replaced by the stone that was moved. The game ends when one player, on their turn, is unable to move, and then that player loses.

Near the end of a game of Clobber, the following position arises:

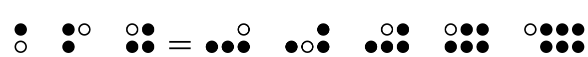


Question 1: If Black goes first, who can win?
Question 2: If White goes first, who can win?
What are the answers to the same questions for the following position?



Discussion: The Clobber positions in these problems can be expressed in terms of the following simpler games:

A *B* *C* *C* *D* *E* *F* *G*



Combinatorial game theory studies games in which the winner is the player who gets the last move. The sum of two games is the game in which each player, at his turn, can make a legal move in any component that he chooses. The negative of a game is the same position with the colors reversed. A game is considered positive if Black can win (no matter who plays next), or negative if White can win (no matter who plays next), or zero if the second player can win (no matter whether that is Black or White). If the second player can win, the value of the game is said to be zero. It is easily seen that adding a zero game to any other game does not change the outcome, because the winner can elect to ignore the zero game except when his opponent plays there, after which he makes the appropriate local response. This ensures that he can get the last move in each component, thereby getting the last move overall.

Any clobber position can be negated by changing the colors of all of the pieces. That's because it is easily seen that second player can win the sum of the game and its negative by following a copy-cat strategy.

The position shown in the first problem is

$$G - B - C$$

The position shown in the second problem is

$$G - B - C - A$$

To solve such problems, it is useful to begin by studying sums and differences of the A's and B's, and how they compare with 0. Taken by itself, A is a quick win for the first player. So it is neither positive nor negative nor 0; it is called FUZZY. But $-A = A$, so $A + A = 0$.

Taken by itself, B is a win for black, so $B > 0$. Therefore, adding B's always increases the value,

$$0 < B < B + B < B + B + B < B + B + B + B < ...$$

The next crucial step is to compare A with multiples of B. From the game $A + B$, the first player can win. If Black moves first, she wins by playing on A, but if White moves first, he wins by playing on B. So $A + B$ is fuzzy, and we write

$$A + B \gtrless 0$$

However, $A + B + B$ is positive. So

$$-B - B < A < B + B$$

even though A is incomparable with $-B$, 0, and B.

When we compare C with sums of A's and B's, we find this startling equality:

$$C = B + B + A.$$

Using this formula, we can formally simplify the second position published in our Emissary column:

$$G - C - B - A = G - B - B - B - A - A = G - B - B - B.$$

So instead of comparing G with $A + B + C$, we could equivalently compare it with the sum of three B's.

In combinatorial game theory, we conventionally express the value of a game by listing the Black and White options separated by a slash, with Black's options on the left and White's on the right. Thus, $E = 0|C, D$. This is because Black has essentially only one option for her next move, which is to end the game immediately, but White has a choice of moving either to C or to D. To compare these choices, we might study the game

$$C - D$$

which is positive. Alternatively, we might compare D with multiples of A's and B's, and discover another surprising equality, namely

$$D = A - B$$

After discovering that, we could write

$$C - D = B + B - A + B > B + B + A > 0$$

In general, Black always prefers larger values, and White always prefers smaller values. So in any environment, from E White should always prefer to move to D rather than to C. Thus,

$$E = 0|D = 0|(A - B)$$

We are now ready to solve the two Clobber problems in our Emissary column. In either case, White should refrain from moving on A or B or C if he can instead move from G or F or E. Black, on the other hand, should refrain from moving on G or F or E if he can instead move on B or C. So on the first three turns, White will transform G to F to E to D. Black will use two of his first three turns to transform $-C$ to $-B$ to A, and the other one to transform the other $-B$ to A. So after each player has made three moves, the value of the first position will be transformed to $D + A + A = D$, which is a win for the next player. So this original position, $G - C - B$, is a win for the first player. On the other hand, after each player has made three moves, the value of the second original position is transformed into $D + A + A + A = D + A = -B$, a win for White. So the second original position is also a win for White no matter who moves first.

In the notation of combinatorial game theory,

$$A = * \text{ (“star”)} \quad B = \uparrow \text{ (“up”)} \quad -B = \downarrow \text{ (“down”)} \quad E = 0|\uparrow * = \uparrow^2 \text{ (“up-second”)}$$

All of these values, and in fact all values which occur in the game of Clobber, are infinitesimals because all sums of them remain smaller than all positive numbers, such as 2^{-n}, which occurs in many other combinatorial games. Although positive, up-second is a higher order infinitesimal than up. F exceeds $B + A$ by an infinitesimal of higher order than up, and similarly, G also exceeds $B + B$ by an infinitesimal of higher order than up. Thus,

$$G - C - B \text{ is slightly larger than } \downarrow *,$$

It is fuzzy, but $G - C - B - A$ is negative.

Problem 3. Find the area of the convex octagon that can be inscribed in a circle and that has four consecutive sides of length 3 and the remaining four consecutive sides of length 2.

Discussion: Reordering the triangles does not change the area of the polygon, and it is especially convenient to alternate the isosceles triangles. A close look at the resulting figure shows that the octagon can be obtained from a square by chopping off the four corners, each being isosceles right triangles of side $\sqrt{2}$ and hypotenuse 2. The side of the square is $3 + 2\sqrt{2}$ and the area of the octagon is

$$(3 + s\sqrt{2})^2 - 4 \cdot (1/2)(\sqrt{2})^2 = 13 + 12\sqrt{2}.$$

Problem 4. A warden meets with 23 prisoners as they arrive. He explains the prison rules, and then allows them to meet each other, talk, and have a strategy session. The prisoners are then taken to their solitary cells, and no further communication between the prisoners is possible.

In the prison there is a "switch room" containing two electrical switches (not connected to anything), each with two positions. From time to time the warden chooses a prisoner and takes him to the switch room; after observing the switches the prisoner is required to choose one switch and reverse its state. (The prisoners know nothing about the initial state of the switches.) The prisoner is then returned to solitary confinement. The only constraint on the warden's behavior is that each prisoner must be taken to the switch room infinitely often, i.e., any individual prisoner knows that he will sooner or later be taken to the switch room, and after that sooner or later taken to the switch room again, etc.

At any time, a prisoner can announce to the warden that "By now, all of us have visited the switch room." If this is a true statement, the prisoners are all freed. If it is false, the prisoners are all executed. Clearly, no prisoner will make this statement unless they are absolutely sure that it is true.

Devise a strategy for the prisoners which will guarantee their release.

Discussion: The prisoners plot their strategy as follows:

1. One of their number is selected as the boss. He will be the only one who will ever declare that everyone has visited the Switching Room. (Being academically trained to be independent, many mathematicians try several strategies based on egalitarian organizations before they are forced to this authoritarian approach.)

2. Only the boss will ever turn switch A `off`, and he'll do so every time he sees it `on`. Every time he sees it `off`, he leaves it unchanged.

3. Every other prisoner will turn switch A from `off` to `on` on his first *two* opportunities to do so. In all other cases, he will leave switch A unchanged.

4. The first time the boss turns switch A `off`, he is uncertain as to whether it began in the `on` state, or whether other prisoners have turned it on exactly once… The 44th time the boss turns switch A `off`, he is uncertain as to whether other prisoners have turned it on exactly 43 times or 44 times. But in either case, he can now safely conclude and declare that every other prisoner has visited the switching room.

Note: Switch B is irrelevant; its only use is as a placebo enabling visitors to leave switch A unchanged when the strategy so dictates. It is of course essential that all prisoners know and agree on the meanings of `on`, `off`, and Switch A.