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Puzzles Column — Solution to Problem 5

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Problem 5

We restate the problem, give a solution, and then describe some of the possible "real-world" interpretations of this circle of ideas.

There is a biased coin whose favored side has probability $u > 1/2$ and unfavored side has probability $v = 1 - u < 1/2$. The direction of the bias is unknown, and the goal of the players of the game is to determine whether heads or tails is favored. For expositional simplicity, we will refer to heads tails as black/white and talk about the color of the favored side.

As in problem 4, there are two teams, one containing n girls and the other containing n boys.

In addition to the biased coin, there are two decks of cards; one consists only of cards that say "report." The other deck has a fraction r "report" cards, and a fraction $s = 1 - r$ of cards that all say either "black" or "white;" the choice is made by a single flip of a (fair) coin by a referee at the beginning. The decks are randomly assigned (by flipping the fair coin again) to the two teams. The black/white cards are called "shill" cards.

No one but the referee knows which team's deck contains the shill cards, nor whether the shills are all heads or all tails.

The game now proceeds as follows. As in Problem 4, the boys and girls line up and each member of the successive couples flips the coin, privately. Each then examines the top card from his or her team's deck. After privately perusing the card, the player returns the card to its deck and reshuffles it. He and she then concurrently announce "heads" or "tails," obeying the following rules:

a. A student with a shill card must ignore all prior information, including the current coin flip and all prior announcements, and must announce the bias specified on the shill card.

b. As before, a student with a "report" card announces the more likely bias of the coin based on all available information, including prior announcements and the single coin flip which he or she has observed.

c. Each then passes its team's deck along to the next member of the same team, and the next couple then behaves according to these same rules.

d. All announcements are public, and all coin flips and cards are private.

Here are some questions to be answered after all $2n$ announcements have been made: (1) What is the probability that all announcements were correct? (2) What is the probability that all announcements were false? (3) What is the probability that there was a complete deadlock: all boys announced one bias and all girls announced the other?

Remarks: As noted earlier, it may be easier to first consider only the limiting case of arbitrarily large n. Also, it may also be simpler to first consider the special case $s = 0$ (Problem 4) and a new special case $s = 1$.

Solution

Let's say an unshilled announcer behaves as an "Observer" if his announcement depends on the color he observes, or as a "Commentator" otherwise.

On Round 1, neither player has any prior information on which to comment, so both behave as observers. Their announcements are given by the following probabilities:

u	*Player on unshilled team is correct*
$\frac{s}{2} + ru$	*Player on shilled team is correct*
v	*Player on unshilled team is incorrect*
$\frac{s}{2} + rv$	*Player on shilled team is incorrect*
$u\left(\frac{s}{2} + ru\right)$	*Both announcements on Round 1 are correct*
$v\left(\frac{s}{2} + rv\right)$	*Both announcements on Round 1 are incorrect*

Now consider a player on Round 2, who receives an unshilled card and observes a black result of a coin flip after hearing both players on Round 1 announce "White." If the coin is biased white, then the probability of this event is $u(s/2 + ru)(v/2 + rv/2)$. If the bias favors black, the probability is $v(s/2 + rv)(u/2 + ru/2)$. The ratio of these is

$$\frac{u(s + 2ru)v(1 + r)}{iv(s + 2rv)u(1 + r)} = \frac{s + 2ru}{s + 2rv} > 1,$$

because $u > v$.

Hence, if the two first round announcements agree, an unshilled player on the 2nd round will Comment rather than Observe. Thus, only shills will disagree with the Round 1 consensus. Hence, the probability that the color of the shills is correct and ALL $2n$ announcements are correct is

$$\frac{u}{2}\left(\frac{s}{2} + ru\right),$$

and the total probability that all $2n$ announcements are correct is

$$\frac{u}{2}\left(\frac{s}{2} + ru\right)\left(1 + r^{n-1}\right).$$

Similarly, the probability that all $2n$ announcements are incorrect is

$$\frac{v}{2}\left(\frac{s}{2} + rv\right)\left(1 + r^{n-1}\right).$$

The simplest case, popularized by economists, has $s = 0$, $r = 1$. In that case, agreement on Round 1 leads to unanimous announcements, which are all correct with probability u^2 and all wrong with probability v^2.

We next suppose that on the first k rounds all boys have announced one color and all girls have announced the other color. Let's further suppose that all have behaved as observers rather than as commentators. Let's further suppose that you are a player on the $(k+1)$-th round, that you draw an unbiased card, and your observed color matches the opposing team's announcements rather than your team's. This can happen in 8 ways, depending on three binary variables: the color towards which the coin is biased, which team is shilled, and the color of the shills. We now tabulate these probabilities. Each entry is a product of 3 factors: Your teammates' announcements; other team's announcements, and my observation, which are listed in that order.

BIAS FAVORS ANNOUNCEMENTS OF WHICH TEAM?	MINE	OPPONENTS'
My team is shilled in my team's announced color	$(s+ru)^k v^k rv$	$(s+rv)^k u^k ru$
My team is shilled in other team's announced color	$(ru)^k v^k rv$	$(rv)^k u^k ru$
Other team is shilled in my team's announced color	$u^k (rv)^k v$	$v^k (ru)^k u$
Other team is shilled in their announced color	$u^k (s+rv)^k v$	$v^k (s+ru)^k u$

Same entries divided by $(ruv)^k$:

My team is shilled in my team's announced color	$\left(\frac{s}{ru}+1\right)^k rv$	$\left(\frac{s}{rv}+1\right)^k ru$
My team is shilled in other team's announced color	rv	ru
Other team is shilled in my team's announced color	v	u
Other team is shilled in their announced color	$\left(\frac{s}{rv}+1\right)^k v$	$\left(\frac{s}{ru}+1\right)^k u$

Whether your announcement will be your observation or your comment depends on which four terms have the larger sum. So let (capital) K be the smallest positive integer (if any exists) for which

$$\left(\frac{s}{rv}+1\right)^K (v-ru) + \left(\frac{s}{ru}+1\right)^K (rv-u) + (r+1)(v-u) > 0.$$

[Notice that except in the degenerate case when $r = v/u$, the first term will become dominant for sufficiently large K, and its sign depends only on $(v - ru)$. Both of the other two terms are necessarily negative.]

We continue to assume that for some $k < K$, the first k pairs of announcements are deadlocked: all boys reporting one color and all girls reporting the other. Then, evidently on the k-th round, both students will continue to announce their observations. However, if $k > K$, then on the k-th round each of the students will Comment in favor of his/her own team, and the deadlock will persist. If the color of the shills is consistent with the announcements of the team which is shilled, then the deadlock will persist for arbitrarily large values of n.

Given this, working out the probability of the complete deadlock then becomes a straightforward calculation, in which the value of K plays a key role.

Verbal Conclusion

Very crudely, one might attempt to give an intuitive explanation of the deadlocking phenomenon as follows: Since I did not draw a shill card, the other team is more likely to be shilled than mine. Hence, their announcements deserve less weight. So the k announcements of my more-likely-unbiased teammates outweigh the k announcements of my opponents, who are more likely to be biased. When k is large enough, this difference in the two consensuses of prior announcements outweighs my own one-marble observation.

If s, the perceived probability of sham, is less than $1 - v/u$, then the risk of deadlock approaches zero as n approaches infinity. Hence, when deadlock occurs in real-life situations, each side's advocates may seek to justify its announcements by imputing increased probability of shamming to their opponents.

Might this discussion help explain smear tactics? The politicization of issues such as climate change? Or the tendency of human committees to reach consensus even on issues where the two sides have nearly equally-compelling cases, and the consensus may well be wrong? Does being the first committee member to speak out give the speaker's announcement more influence than it deserves?

How often do human announcers represent themselves as observers, even when they are merely commentators? And how many jury members realize that most "expert witnesses," as well as most trial lawyers, are often really acting only as shills?