SOLUTION TO PROBLEM 4B
SPRING/SUMMER 2010 EMISSARY

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Theorem. *If* $N \leq 32$*, then there exists a unique necklace of nodes* 1 *through* N *connected by squares.*

Proof. If $N \geq 2$, then 2 requires 7 and 14 (or greater).
If $N \geq 8$, then 8 requires 1 and 17.
If $N \geq 16$, then 16 requires 9 and 20.
If $N \geq 18$, then 18 requires 7 and 31.
Hence $N \geq 31$, so we need consider only $N = 31$ and $N = 32$.

So we search for all possible necklaces (also called "loops") in the graph whose 32 nodes are labelled with integers from 1 to 32.

As seen in Figure 1, 25 is adjacent to only 24 and 11, so both of these branches must be used. Similarly, 28 must connect to both 21 and 8. Node 30 must connect to both 6 and 19. Although there is a possible branch from 6 to 19, using it would create a triangle isolated from the rest of the graph, so 6∼19, by which we mean that the branch between 6 and 19 cannot be used in the necklace. Node 27 must connect to 22 and 9, and 16 must connect to 9 and 20. Hence 9 connects to both 27 and 16, and hence cannot connect to 7. The rest of Figure 1 follows by similar arguments. Since the paths through the nodes inside the dotted region of Figure 1 are determined, all other branches in the necklace must lie outside of the dotted region. If it is used, node 32 must lie between 17 and 4. But if $N = 31$, node 32 cannot be used. So we treat 17–4 as a special optional branch. If it is used in the necklace, then $N = 32$; if not, then $N = 31$.

Figure 2 shows all of the possible branches outside of the dotted region in Figure 1. Since the boxed nodes are terminals of paths shown in Figure 1, in Figure 2 they must have valence 1. The circled nodes in Figure 2, which have no branches in Figure 1, must have valence 2 in Figure 2. We shall now see that these constraints admit only 5 solutions within Figure 2.

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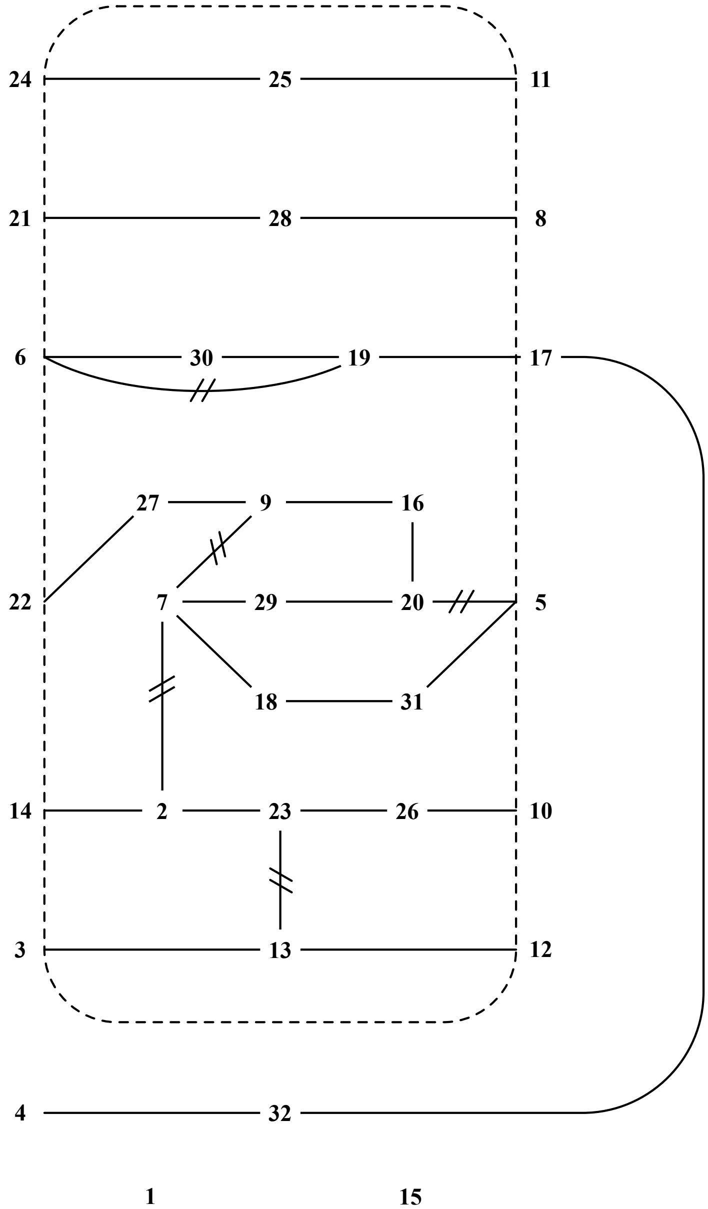

FIGURE 1. Forced paths inside dotted box. (All branches outside dotted box except 4–32–17 are not shown.)

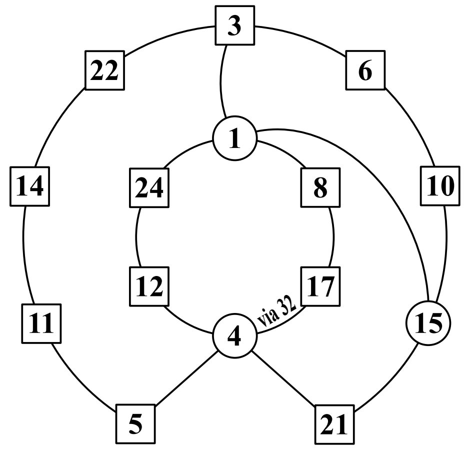

FIGURE 2. Complement of branches inside dotted box of Fig. 1. Circled nodes must have valence 2; squared nodes must have valence 1. Note 32 is absorbed into the "branch" from 17 to 4.

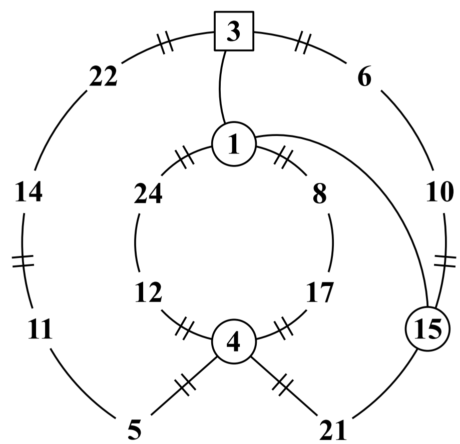

FIGURE 3. 1–3 forces 15–1 and 15–21, leading to isolation of node 4.

Let's now assume the necklace uses the branch 1–3. As shown in Figure 3, this forces 3∼6, 6–10, 10∼15, and hence 15–1 and 15–21. Similarly, 1–3 forces 3∼22, etc., eventually leading to isolation of node 4. So we conclude that 1∼3.

We then assume that 1∼15. This forces the unique solution (of Figure 2) shown in Figure 4.

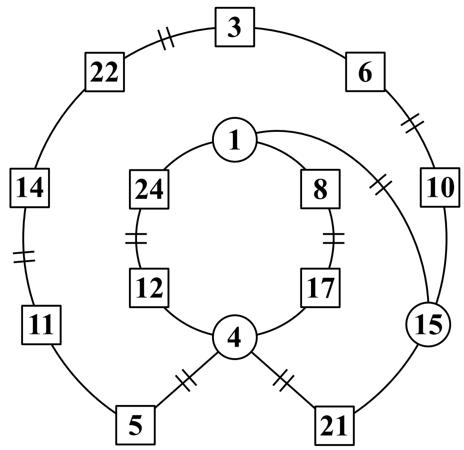

FIGURE 4. 1∼15 forces unique solution.

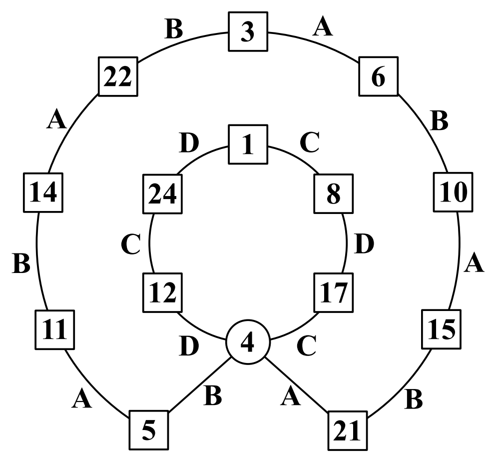

FIGURE 5. 1–15 has four solutions.

Finally, we assume that 1–15. This reduces Figure 2 to Figure 5. In this figure, we must connect either all A's or all B's, and all C's or all D's. These 4 solutions of Figure 5, following the unique solution of Figure 4, and each of their completions using Figure 1, are shown in Figure 6.

All five of the "solutions" in Figure 6 partition the nodes from 1 to N into a sum of necklaces. But in 4 of the 5 cases, the partition contains 2 disjoint necklaces. AC is the unique solution in which all nodes form a single connected necklace. Since it uses the "branch" from 4 to 17 via 32, we conclude that there is no necklace with $N = 31$ and that the necklace with $N = 32$ is unique.

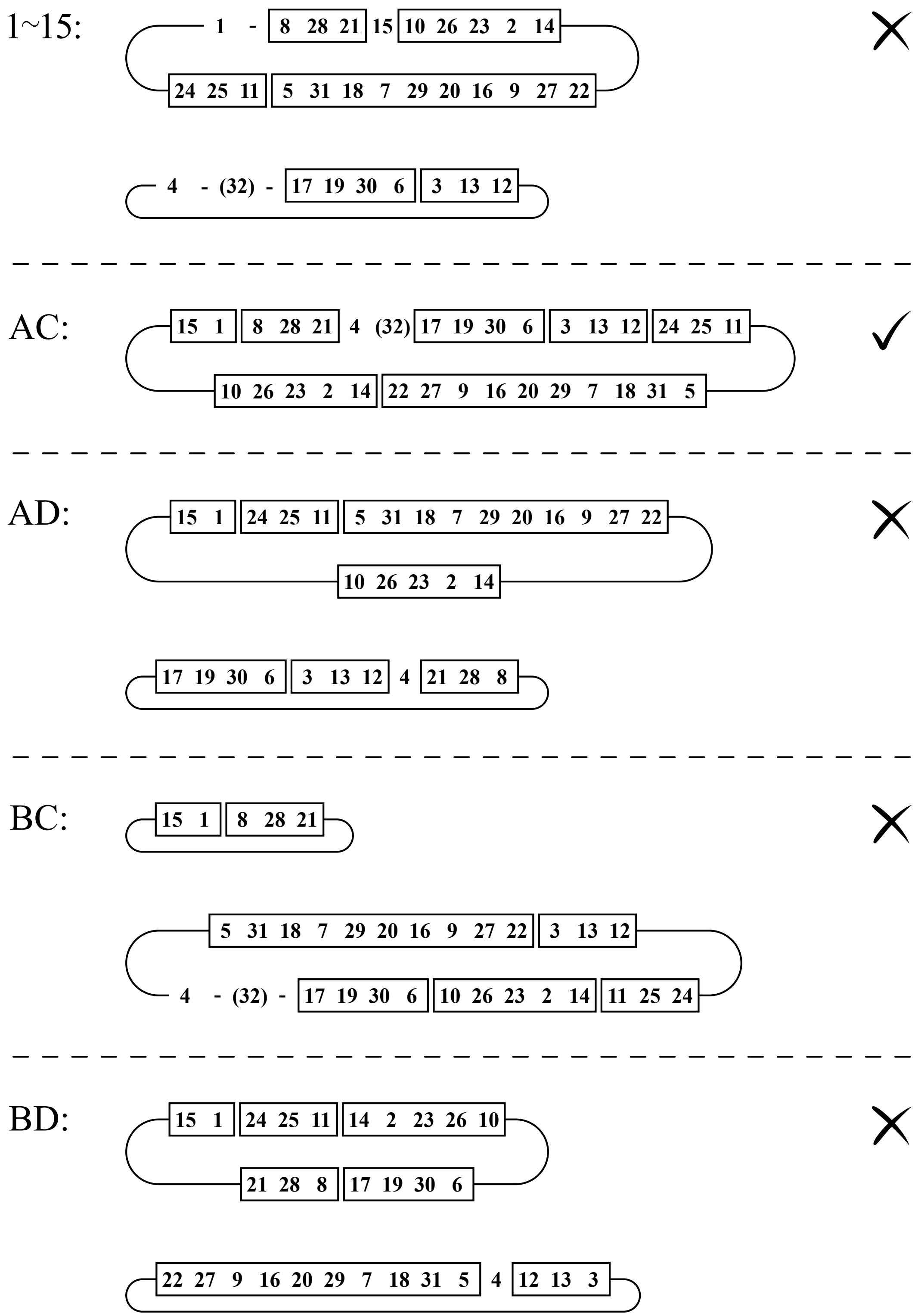

Figure 6. Combinations of solutions of Figure 2 with Figure 1