# Lecture 3. A Proof of Ihara's Theorem, Edge & Path Zetas, Connections with Quantum Chaos

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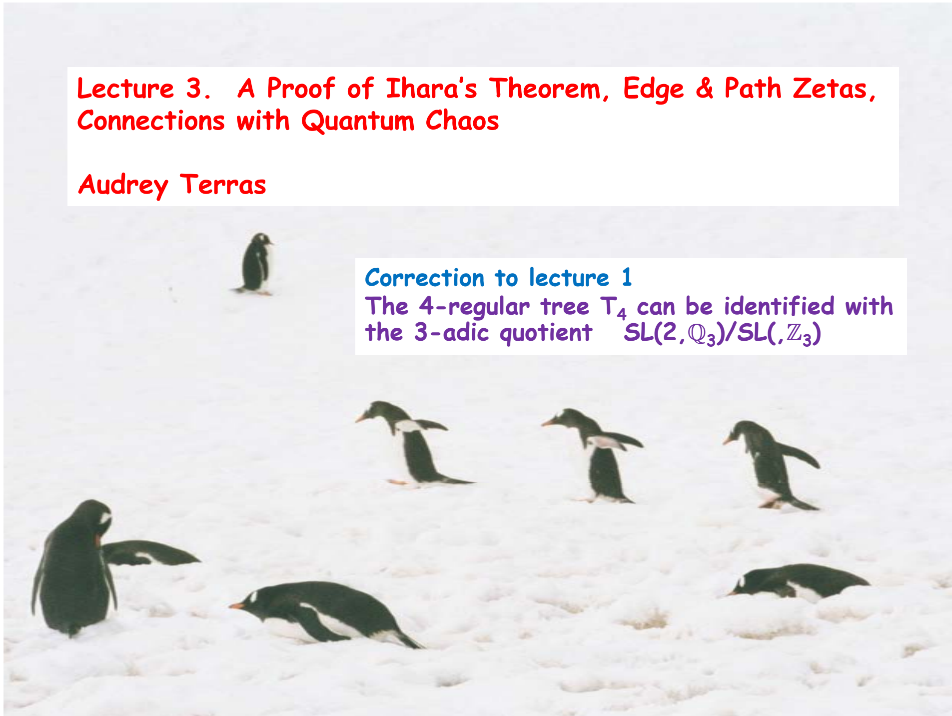
**Correction to lecture 1**
The 4-regular tree $T_4$ can be identified with the 3-adic quotient $SL(2,\mathbb{Q}_3)/SL(,\mathbb{Z}_3)$



## Ihara Zeta Function

$$\zeta(u,X)=\prod_{\substack{[C]\\ \text{prime}}}\left(1-u^{\nu(C)}\right)^{-1}$$

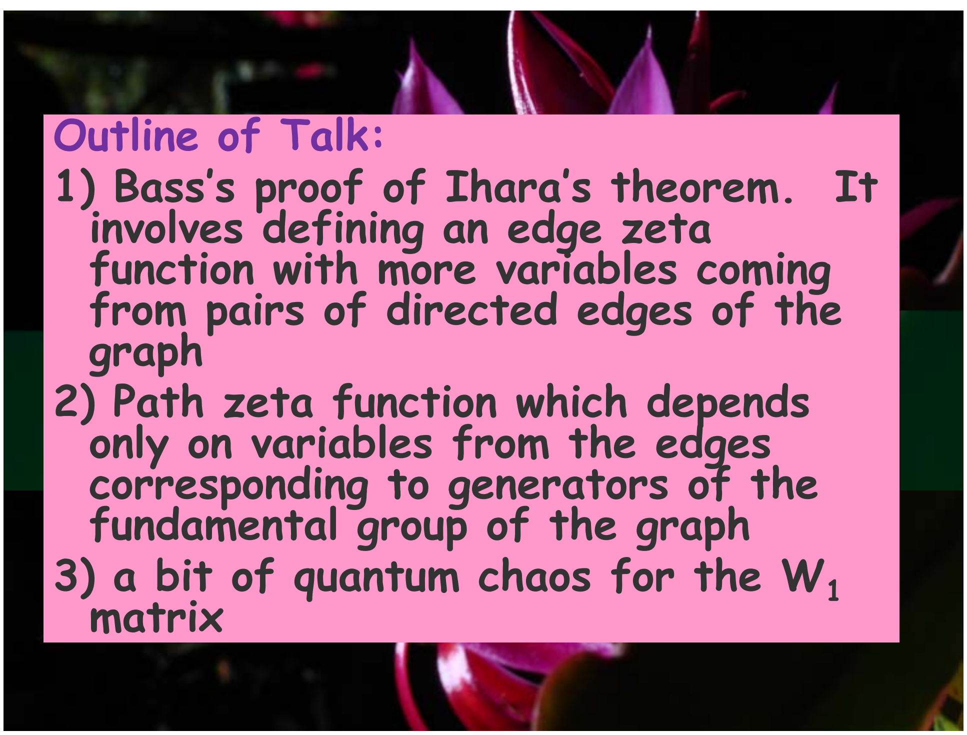
$\nu(C)$ = # edges in $C$
converges for u complex, |u| small

**Ihara's Theorem.**

$$\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$$

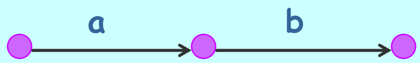
A=adjacency matrix, Q +I = diagonal matrix of degrees,
r=rank fundamental group.


Basic Assumptions
graphs are
connected,
with r=rank fundamental group > 1,
no degree 1 vertices (called leaf vertex, hair, danglers, ...)


Outline of Talk:
1) Bass's proof of Ihara's theorem. It involves defining an edge zeta function with more variables coming from pairs of directed edges of the graph
2) Path zeta function which depends only on variables from the edges corresponding to generators of the fundamental group of the graph
3) a bit of quantum chaos for the $W_1$ matrix

## Edge Zetas

Orient the edges of the graph. Multiedge matrix W has ab entry $w_{ab}$ in $\mathbb{C}$, $w(a,b)=w_{ab}$ if the edges a and b look like




and a is not the inverse of b

Otherwise set $w_{ab}=0$.

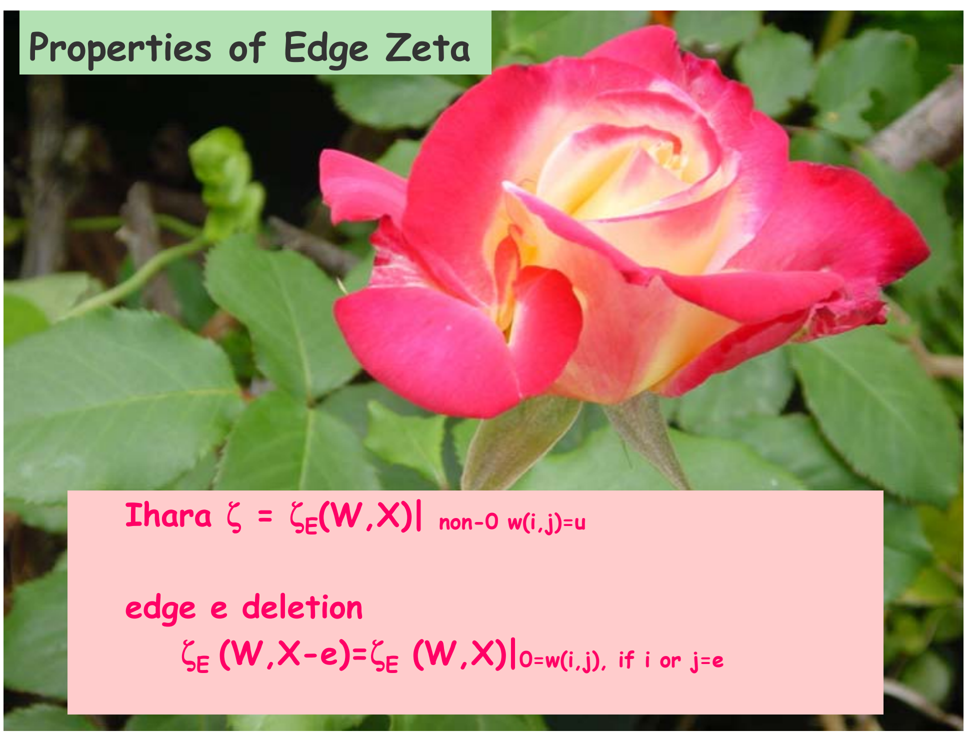
For a prime $C = a_1 a_2 \ldots a_s$, define the edge norm

$$N_E(C) = w(a_s,a_1)w(a_1,a_2)w(a_2,a_3)\cdots w(a_{s-1},a_s)$$

Define the edge zeta for small $|w_{ab}|$ as

$$\zeta_E(W,X) = \prod_{[C]} \left(1 - N_E(C)\right)^{-1}$$

## Properties of Edge Zeta



Ihara $\zeta = \zeta_E(W,X)|_{\text{non-0 } w(i,j)=u}$

edge e deletion

$\zeta_E(W,X-e) = \zeta_E(W,X)|_{0=w(i,j),\text{ if } i \text{ or } j=e}$

# Determinant Formula

## For Edge Zeta

$$\zeta_E(W,X) = \det\left(I - W\right)^{-1}$$

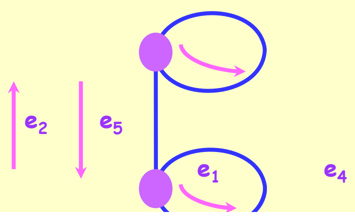
From this Bass gives an ingenious proof of Ihara's theorem.

Reference:

Stark and T., *Adv. in Math.*, Vol. 121 and 154 and 208 (1996 and 2000 and 2007)

## Example

D=Dumbbell Graph



## Part 2 of Bass Proof

$W_1$ matrix obtained from W by setting all non-zero $w_{ij}$ equal to 1

$W_1 + J = T^t S$, where J compensates
for not allowing edge $e_j$ to feed into $e_{j\pm|E|}$

Below all matrices are (|V|+2|E|) x (|V|+2|E|), with |V| x |V| 1st block.

The preceding formulas imply that:

$$\begin{pmatrix} I_{|V|} & 0 \\ T^t & I_{2|E|} \end{pmatrix} \begin{pmatrix} I_{|V|}(1-u^2) & Su \\ 0 & I_{2|E|} - W_1 u \end{pmatrix}$$
$$= \begin{pmatrix} I_{|V|} - Au + Qu^2 & Su \\ 0 & I_{2|E|} + Ju \end{pmatrix} \begin{pmatrix} I_{|V|} & 0 \\ T^t - S^t u & I_{2|E|} \end{pmatrix}$$

Then take determinants of both sides to see

$$(1-u^2)^{|V|} \det(I_{2|E|} - W_1 u) = \det(I_{|V|} - Au + Qu^2)\det(I_{2|E|} + Ju)$$

## End of Bass Proof

$$(1-u^2)^{|V|} \det(I_{2|E|} - W_1 u) = \det(I_{|V|} - Au + Qu^2)\det(I_{2|E|} + Ju)$$

$$I + Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix} \text{ implies } \begin{pmatrix} \mathrm{I} & 0 \\ \text{-Iu} & \mathrm{I} \end{pmatrix}(I + Ju) = \begin{pmatrix} I & Iu \\ 0 & I(1-u^2) \end{pmatrix}$$

So $\det(I+Ju)=(1-u^2)^{|E|}$

**Since r-1=|E|-|V|, for a connected graph, the Ihara formula for the vertex zeta function follows from the edge zeta determinant formula.**

Next we define a zeta function invented by Stark which has several advantages over the edge zeta.

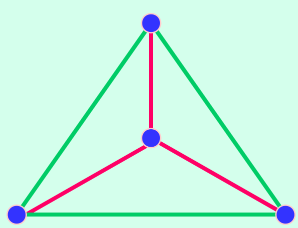
It can be used to compute the edge zeta using smaller determinants.

It gives the edge zeta for a graph in which an edge has been fused.

## spanning trees

A tree is a connected graph without cycles.

A spanning tree for a graph X is a subgraph which is a tree and which contains all the vertices of X.



the red graph is a spanning tree for $K_4$

# Path Zeta Function

**Fundamental Group** of X can be identified with group generated by edges left out of a spanning tree $e_1,\ldots e_r, e_1^{-1},\ldots,e_r^{-1}$

2r ×2r **multipath matrix** Z has ij entry

$z_{ij}$ in $\mathbb{C}$ if $e_j \neq e_i^{-1}$, $z_{ij}$ =0, otherwise.

Imitate the definition of the edge zeta function.

Define for a prime path

$$C = a_1 \cdots a_s,\ \ where\ \ a_j \in \{e_1^{\pm 1},\ldots,e_r^{\pm 1}\}$$

the **path norm** $N_P(C) = z(a_s,a_1)\prod_{i=1}^{s-1} z(a_i,a_{i+1})$

**Define the path zeta function**

$$\zeta_P(Z,X) = \prod_{[C]}\left(1-N_P(C)\right)^{-1}$$

# Specialize Path Zeta to Edge Zeta

**edges left out of a spanning tree** T of X are $e_1,\ldots e_r$

inverse edges are $e_{r+1} = e_1^{-1},\ldots,e_{2r} = e_r^{-1}$

**edges of the spanning tree** T are $t_1,\ldots,t_{|X|-1}$

with inverse edges $t_{|X|},\ldots t_{2|X|-2}$

If $e_i \neq e_j^{-1}$, write the part of the path between $e_i$ and $e_j$ as the (unique) product $t_{k_1}\cdots t_{k_n}$

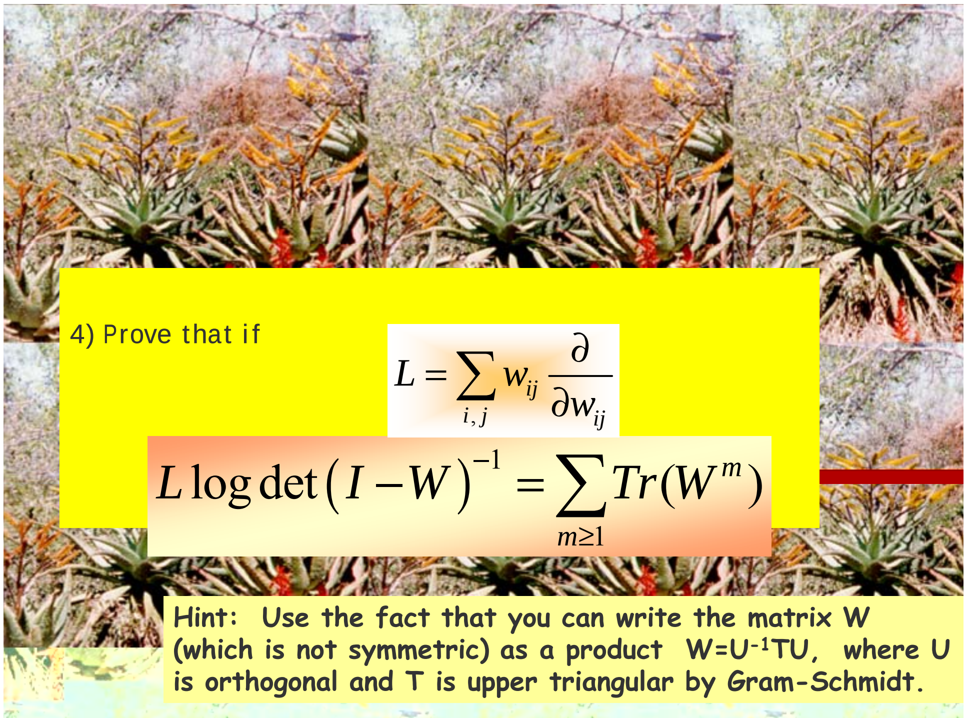
A prime cycle C is first written as a reduced product of generators of the fundamental group $e_j$ and then a product of actual edges $e_j$ and $t_k$.

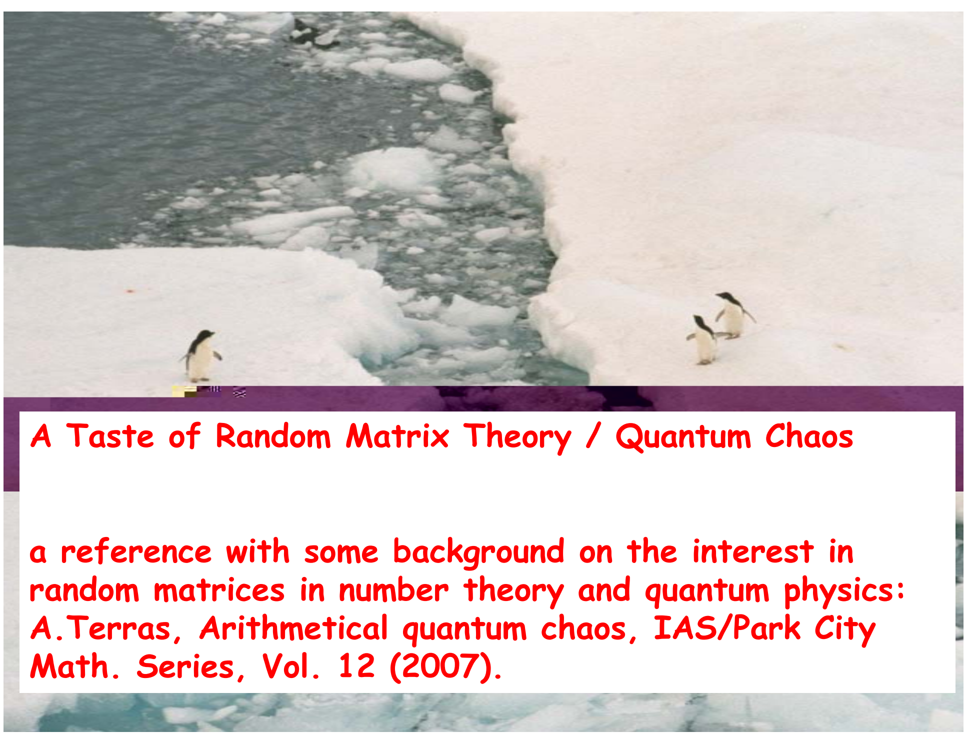
Now **specialize the multipath matrix Z to Z(W)** with entries

Then $$z_{ij} = w(e_i, t_{k_1})w(t_{k_n}, e_j)\prod_{v=1}^{n-1} w(t_{k_v}, t_{k_{v+1}})$$

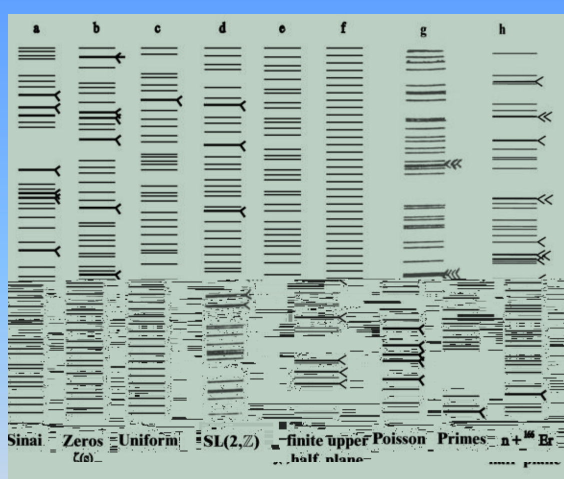
$$\zeta_P(Z(W),X) = \zeta_E(W,X)$$

# Example - the Dumbbell


4) Prove that if
$$L = \sum_{i,j} w_{ij} \frac{\partial}{\partial w_{ij}}$$
$$L \log \det (I - W)^{-1} = \sum_{m \geq 1} Tr(W^m)$$
Hint: Use the fact that you can write the matrix W (which is not symmetric) as a product $W=U^{-1}TU$, where U is orthogonal and T is upper triangular by Gram-Schmidt.


A Taste of Random Matrix Theory / Quantum Chaos
a reference with some background on the interest in random matrices in number theory and quantum physics: A.Terras, Arithmetical quantum chaos, IAS/Park City Math. Series, Vol. 12 (2007).

In lecture 1 we mentioned the experimental connections between statistics of spectra of random symmetric real matrices and the statistics of imaginary parts of s at poles of Ihara $\zeta(q^{-s})$ (analogous to statistics of imaginary parts of zeros of Riemann $\zeta$ and spectra of Hermitian matrices).

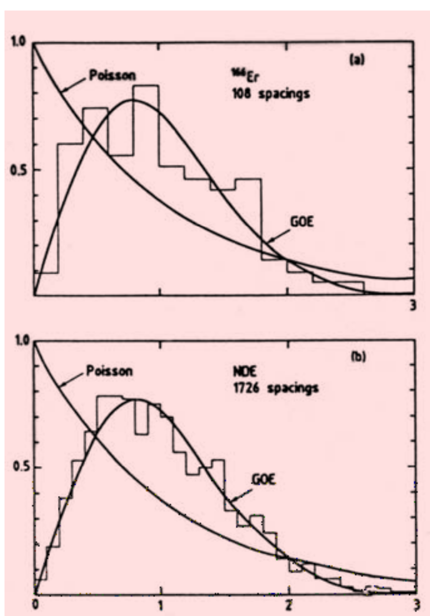



from O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984:

arrows mean lines are too close to distiguish

O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984: "The question now is to discover the stochastic laws governing sequences having very different origins, as illustrated in" the Figure, each column with 50 levels ..." Note that the spectra have been rescaled to the same vertical axis from 0 to 49.

(a) Poisson spectrum, i.e., of a random variable with spacings of probability density $e^{-x}$.
(b) primes between 7791097 and 7791877.
(c) resonance energies of compound nucleus observed in the reaction n+166Er.
(d) from eigenvalues corresponding to transverse vibrations of a membrane whose boundary is the Sinai billiard which is a square with a circular hole cut out centered at the center of the square.
(e) the positive imaginary parts of zeros of the Riemann zeta function (from the 1551th to the 1600th zero).
(f) is equally spaced - the picket fence or uniform distribution.
(g) from P. Sarnak, Arithmetic quantum chaos, *Israel Math. Conf. Proc.*, 8 (1995), (published by Amer. Math. Soc.) : eigenvalues of the Poincaré Laplacian on the fundamental domain of the modular group $SL(2,\mathbb{Z})$, 2 × 2 integer matrices of determinant 1.
(h) spectrum of a finite upper half plane graph for p=53 ($a = \delta = 2$), without multiplicity (see my book *Fourier Analysis on Finite Groups*)



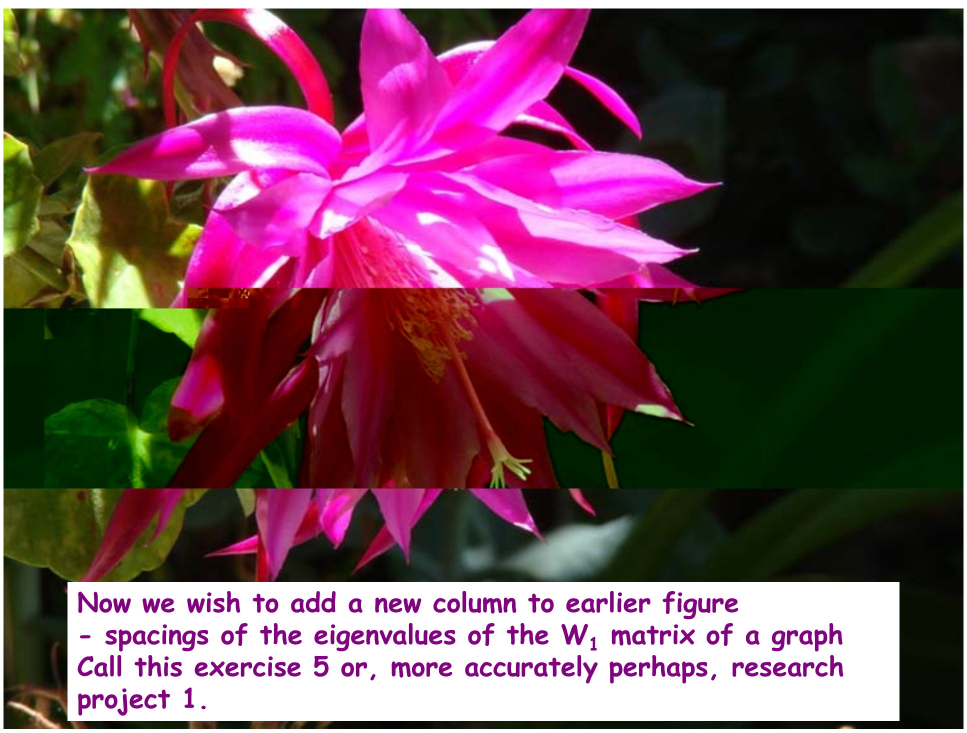

The Figure is from from Bohigas, Haq, and Pandey, Fluctuation properties of nuclear energy levels and widths: comparison of theory with experiment, in K.H. Bockhoff (Ed.), *Nuclear Data for Science and Technology*, Reidel, Dordrecht, 1983)
Level spacing histogram for
(a) 166Er and
(b) a nuclear data ensemble.

**Wigner surmise for spacings of spectra of random symmetric real matrices**

This means that you arrange the eigenvalues) $E_i$ in decreasing order: $E_1 \geq E_2 \geq \cdot \quad \cdot \quad \cdot \geq E_n$. Assume that the eigenvalues are normalized so that the mean of the level spacings $|E_i - E_{i+1}|$ is 1.

**Wigner's Surmise** from 1957 says the level (eigenvalue) spacing histogram is ≈ the graph of the function $\frac{1}{2}\pi x \exp(-\pi x^2/4)$ ,if the mean spacing is 1. In 1960, Gaudin and Mehta found the correct distribution function which is close to Wigner's. The correct graph is labeled GOE in the Figure preceding. Note the level repulsion indicated by the vanishing of the function at the origin. Also in the preceding Figure, we see the Poisson density which is $e^{-x}$.

A reference is Mehta, *Random Matrices*.



Here although $W_1$ is not symmetric, the nearest neighbor spacing (i.e., histogram of minimum distances between eigenvalues) is also of interest.

$\nexists$ many references on the study of spacings of spectra of non-Hermitian or non-symmetric matrices. I did find one: P. LeBoef, Random matrices, random polynomials, and Coulomb systems.
He studies the ensemble of matrices introduced by J. Ginibre, J. Math. Phys. 6, 440 (1965).

An approximation to the distribution of spacings of eigenvalues of a complex matrix (analogous to the Wigner surmise for Hermitian matrices) is:

$$4\Gamma\left(\frac{5}{4}\right)^4 s^3 e^{-\Gamma\left(\frac{5}{4}\right)^4 s^4}$$

Since our matrix is real, this will probably not be the correct Wigner surmise.

I haven't done this experiment yet. In what follows, I just plot the reciprocals of the eigenvalues of $W_1$ - the poles of Ihara zeta for various graphs. The distribution looks rather different than that of a random real matrix with the properties of $W_1$.

## Statistics of the poles of Ihara zeta or reciprocals of eigenvalues of the Edge Matrix $W_1$

**Define** $W_1$ to be the 0,1 matrix you get from W by setting all non-0 entries of W to be 1.

**Theorem.** $\zeta(u,X)^{-1}=\det(I-W_1u)$.

**Corollary.** The poles of Ihara zeta are the reciprocals of the eigenvalues of $W_1$.

The pole R of zeta is:

R=1/Perron-Frobenius eigenvalue of $W_1$.

## Properties of $W_1$

1) $W_1 = \begin{pmatrix} A & B, \\ C & A^T \end{pmatrix}$ B and C symmetric real, A real

2) Row sums of entries are $q_j+1$=degree vertex which is start of edge j.

## Poles Ihara Zeta are in region $q^{-1} \le R \le |u| \le 1$,

q+1=maximum degree of vertices of X.

## Theorem of Kotani and Sunada

If p+1=min vertex degree, and q+1=maximum vertex degree, non-real poles u of zeta satisfy

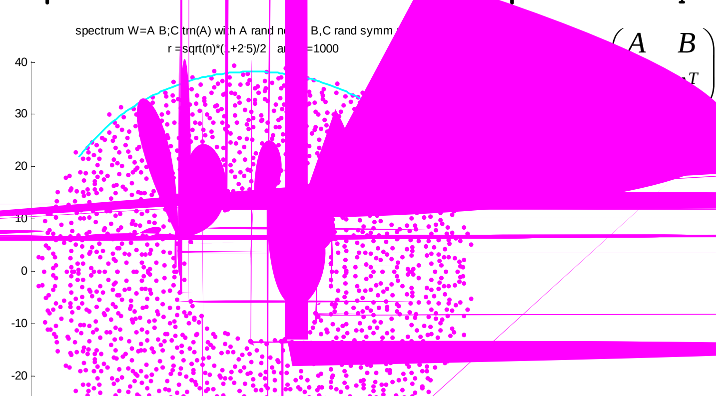
$$\frac{1}{\sqrt{q}} \le |u| \le \frac{1}{\sqrt{p}}$$

Kotani & Sunada, *J. Math. Soc. U. Tokyo*, 7 (2000)

or see my manuscript on my website:

www.math.ucsd.edu\~aterras\newbook.pdf

### Spectrum of Random Matrix with Properties of $W_1$-matrix



# What is the meaning of the RH for irregular graphs?

For irregular graph, natural change of variables is $u=R^s$, where

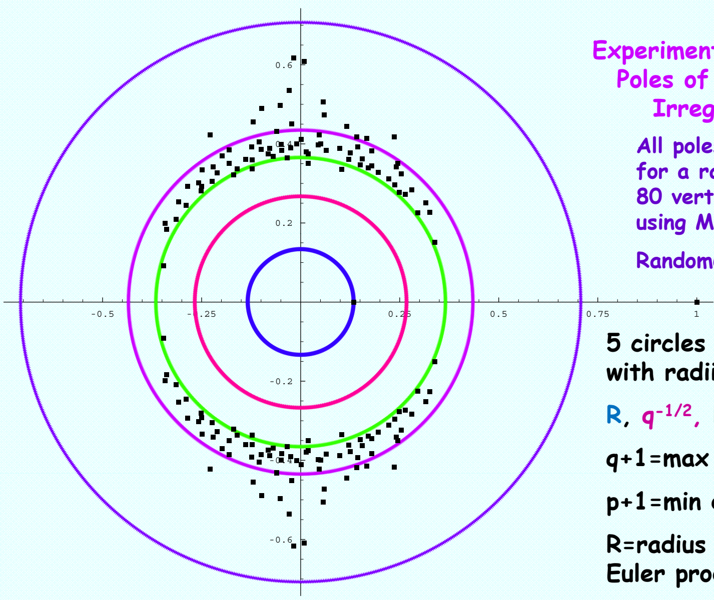
R = radius of convergence of Dirichlet series for Ihara zeta.

Note: R is closest pole of zeta to 0. No functional equation.

Then the critical strip is $0\le \mathrm{Re}\, s\le 1$ and translating back to u-variable. In the q+1-regular case, R=1/q.

**Graph theory RH:**

$\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$




**Experiment on Locations of Poles of Ihara Zeta of Irregular Graphs**

All poles but -1 of $\zeta_X(u)$ for a random graph with 80 vertices denoted by ■ using Mathematica:

RandomGraph[80,1/10]

5 circles centered at 0 with radii

$R$, $q^{-1/2}$, $R^{1/2}$, $(pq)^{-1/4}$, $p^{-1/2}$

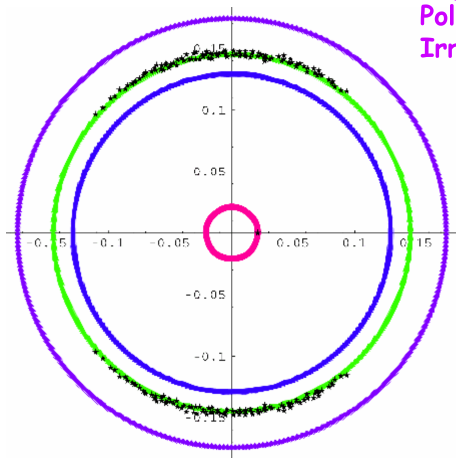
q+1=max degree,

p+1=min degree,

R=radius of convergence of Euler product for $\zeta_X(u)$

RH is false but poles are not far inside circle of radius $R^{1/2}$

RandomGraph[80,1/10] means probability of edge between 2 vertices =1/10.




## Experiment on Locations of Poles of Ihara Zeta of Irregular Graphs

All poles except -1 of $\zeta_X(u)$ for a random graph with 100 vertices are denoted ■, using Mathematica

RandomGraph[100,1/2]

Circles centered at 0 with radii

R, $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$

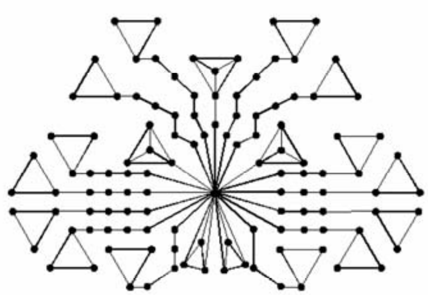
q+1=max degree,

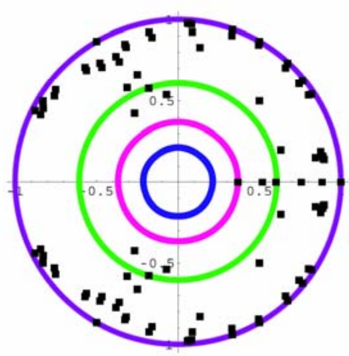
p+1=min degree

R=radius of convergence of product for $\zeta_X(u)$

RH is false maybe not as false as in previous example with probability 1/10 of an edge rather than ½.

Poles clustering on RH circle (green)



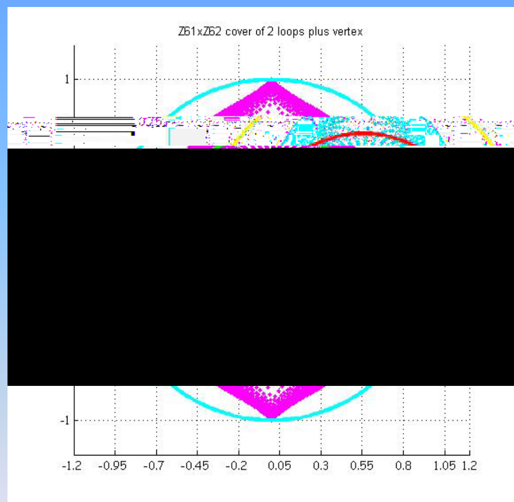


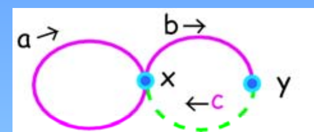

Matthew Horton's Graph has 1/R ≅ e to 7 digits.

Poles of Ihara zeta are boxes on right. Circles have radii $R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}$, if q+1=max deg, p+1=min deg. Here

The RH is false. Poles more spread out over plane.

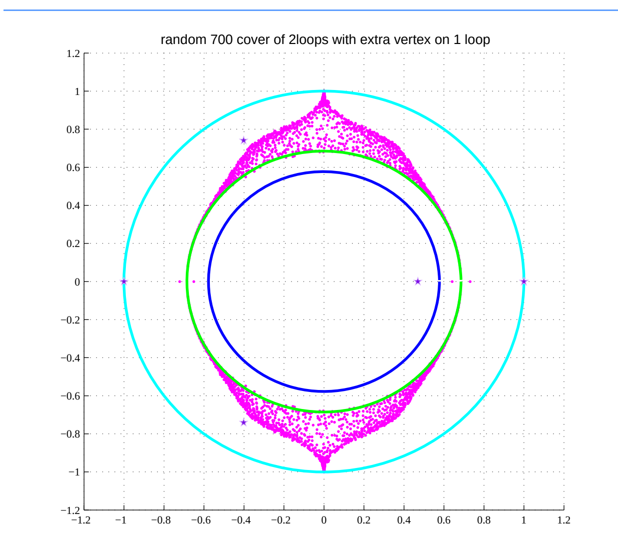
## Poles of Ihara Zeta for a $\mathbb{Z}_{61}\times\mathbb{Z}_{62}$-Cover of 2 Loops + Extra Vertex are pink dots

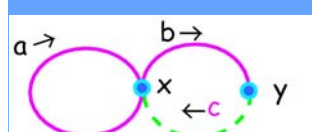


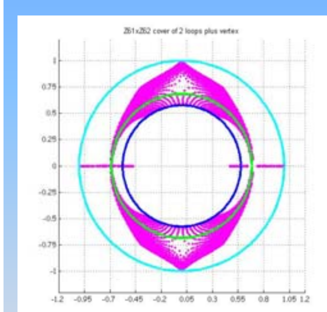




Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$ ,1; $R \cong .47$

**RH very False**



Z is random 700 cover of 2 loops plus vertex graph in picture.

The pink dots are at poles of $\zeta_Z$. Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with $q=3$, $p=1$, $R \cong .4694$. RH approximately True.

References: 3 papers with Harold Stark in *Advances in Math*.

- Paper with Matthew Horton & Harold Stark in Snowbird Proceedings, Contemporary Mathematics, Volume 415 (2006)
  Quantum Graphs and Their Applications, *Contemporary Mathematics*, v. 415, AMS, Providence, RI 2006.
- See my draft of a book:
  www.math.ucsd.edu/~aterras/newbook.pdf
- Draft of new paper joint with Horton & Stark: also on my website
  www.math.ucsd.edu/~aterras/cambridge.pdf
- There was a graph zetas special session of this AMS meeting – many interesting papers some on my website.
- For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications,* 425 (2007) 130–142.
- work of Angel, Friedman and Hoory giving analog of Alon conjecture for irregular graphs, implying our Riemann Hypothesis (see Joel Friedman's website: www.math.ubc.ca/~jf)