Modular Forms in Vertex Operator Algebras

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Abstract

We discuss the way in which modular forms and elliptic functions naturally arise in the theory of Vertex Operator Algebras (VOAs). We consider Zhu recursion formulas for these modular forms and explicitly compute examples in the Heisenberg and lattice VOAs. The discussion is informal in style but reasonably self-contained.

1 Elliptic Functions and Modular Forms

Some notation: $\mathbf{Z}$ is the set of integers, $\mathbf{R}$ the real numbers, $\mathbf{C}$ the complex numbers, $\mathbf{H}$ the complex upper-half plane. We will always assume $\tau \in \mathbf{H}$ and $z \in \mathbf{C}$ unless otherwise noted. For a symbol z we set $q_z = \exp(z)$ and $q = q_{2\pi i\tau} = \exp(2\pi i\tau)$.

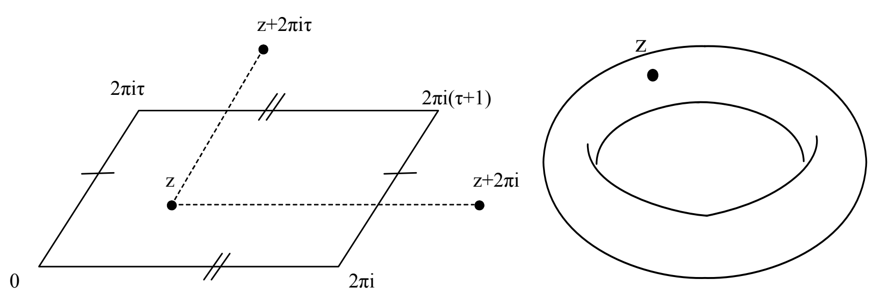
We discuss a number of modular and elliptic functions that we will need. We begin with the classical elliptic Weierstrass $\wp$-function [La]

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{m,n\in\mathbf{Z}}{}' [\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2}]. \tag{1}$$

for $(z,\tau) \in \mathbf{C}\times\mathbf{H}$ with $\omega_{m,n} = 2\pi i(m\tau + n)$ and where the prime indicates that $(m,n) \neq (0,0)$. The double sum is absolutely convergent and hence independent of the order of summation. $\wp$ is analytic in $z \in \mathbf{C}$ (with a 2nd order pole at $z = \omega_{m,n}$ for all $m,n \in \mathbf{Z}$) and is periodic with periods $2\pi i$ and $2\pi i\tau$ i.e.

$$\wp(z + 2\pi i, \tau) = \wp(z + 2\pi i\tau, \tau) = \wp(z,\tau). \tag{2}$$

Thus $\wp$ is an elliptic function i.e. an analytic function on the torus (elliptic curve) $\mathbf{C}/\Lambda = \{z|z \sim z + \omega_{m,n} \text{ for all } m,n\}$ where $\Lambda = \{\omega_{m,n}\}$ denotes the complex lattice generated by the basis $2\pi i$ and $2\pi i\tau$. We may represent the torus by the fundamental parallelogram shown with identified sides



Note that for z in the fundamental parallelogram $-2\pi\,\mathrm{Im}\,\tau < \mathrm{Re}(z) < 0$ so that $|q| < |q_z| < 1$.

There is a natural action of $SL(2,\mathbf{Z})$ on $\mathbf{C}\times\mathbf{H}$ given for all $\gamma \in SL(2,\mathbf{Z})$ by

$$\gamma : z \mapsto (\gamma.z, \gamma.\tau) = (\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}), \tag{3}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. This corresponds to a basis change $2\pi i(\tau, 1) \rightarrow 2\pi i(a\tau + b, c\tau + d)$ for Λ followed by a conformal rescaling $z \rightarrow z/(c\tau + d)$. Then it follows that

$$\wp(\gamma.z, \gamma.\tau) = (c\tau + d)^2 \wp(z, \tau). \tag{4}$$

Considering the modular transformation $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we obtain $\wp(z, \tau + 1) = \wp(z, \tau)$. Together with the elliptic periodic property $\wp(z + 2\pi i, \tau) = \wp(z, \tau)$ it follows that $\wp(z, \tau)$ has a Fourier expansion in both q and q_z. To describe this we firstly define

$$P_1(z, \tau) = \sum_{n \in \mathbf{Z}}' \frac{q_z^n}{1 - q^n} - \frac{1}{2}, \tag{5}$$

$$P_2(z, \tau) = \frac{\partial}{\partial z} P_1(z, \tau) = \sum_{n \in \mathbf{Z}}' \frac{n q_z^n}{1 - q^n}, \tag{6}$$

where here the prime indicates that $n \neq 0$. The factor of $-1/2$ in (5) ensures that $P_1(z, \tau)$ is odd in z. $P_1(z, \tau)$ and its derivatives are absolutely convergent for $|q| < |q_z| < 1$. We next show

Proposition 1.1.

$$P_2(z, \tau) = \wp(z, \tau) + E_2(\tau), \tag{7}$$

for

$$E_2(\tau) = \sum_{m \in \mathbf{Z}}' \left[\sum_{n \in \mathbf{Z}}' \frac{1}{\omega_{m,n}^2} \right], \tag{8}$$

where the prime indicates that $(m, n) \neq (0, 0)$.

Remark 1.2. *The nested double sum is* not *absolutely convergent so that sum depends on the order of summation.*

We prove Proposition 1.1 we firstly note the identity

$$\sum_{n \in \mathbf{Z}} \frac{1}{(x - 2\pi i n)^2} = \frac{q_x}{(1 - q_x)^2}. \tag{9}$$

Exercise 1.3. *Verify (9) by comparing the pole structures.*

We next note that

$$\wp(z,\tau)+E_2(\tau)=\sum_{m\in\mathbf{Z}}\left[\sum_{n\in\mathbf{Z}}\frac{1}{(z-\omega_{m,n})^2}\right],$$

(where the convergent nested double sum depends on the order of summation). Apply (9) for $x=z-2\pi i m\tau$ with $q_x=q_zq^{-m}$ so that

$$\wp(z,\tau)+E_2(\tau)=\sum_{m\in\mathbf{Z}}\frac{q_zq^m}{(1-q_zq^m)^2}.$$

The RHS can be rewritten as

$$\frac{q_z}{(1-q_z)^2}+\sum_{m>0}\left(\frac{q_zq^m}{(1-q_zq^m)^2}+\frac{q_zq^{-m}}{(1-q_zq^{-m})^2}\right).$$

Since $|q_zq^m|,|q_z^{-1}q^m|<1$ for $m>0$ this is

$$\begin{aligned}&\frac{q_z}{(1-q_z)^2}+\sum_{m>0}\sum_{n>0}n(q_z^n+q_z^{-n})q^{nm}\\ =\;&\frac{q_z}{(1-q_z)^2}+\sum_{n>0}n\left(q_z^n+q_z^{-n}\right)\frac{q^n}{1-q^n}.\end{aligned}\tag{10}$$

Rearranging the summands we obtain

$$\begin{aligned}&\sum_{n>0}n\left((1+\frac{q^n}{1-q^n})q_z^n+\frac{q^n}{1-q^n}q_z^{-n}\right)\\ =\;&\sum_{n>0}n\left(\frac{q_z^n}{1-q^n}-\frac{q_z^{-n}}{1-q^{-n}}\right)=P_2(z,\tau),\end{aligned}$$

as claimed. ∎

Consider the expansion of P_2 in z

$$P_2(z,\tau)=\frac{1}{z^2}+\sum_{k\geq 2}(k-1)E_k(\tau)z^{k-2},$$

where from (1) we find

$$E_k(\tau)=\sum_{m,n\in\mathbf{Z}}{}^{'}\frac{1}{\omega_{m,n}^k}=\frac{1}{(2\pi i)^k}\sum_{m,n\in\mathbf{Z}}{}^{'}\frac{1}{(m\tau+n)^k}.\tag{11}$$

$E_k(\tau) = 0$ for k odd, and for $k \geq 4$ even is called the Eisenstein series [Se]. (Note that other normalizations of E_k appear in the literature!). Expanding (10) we obtain the Fourier expansion for Eisenstein series for $k \geq 2$ even

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}, \tag{12}$$

where B_k is the kth Bernoulli number given by

$$\frac{q_z}{(1-q_z)^2} = \frac{1}{z^2} - \sum_{k \geq 2, \text{even}} \frac{B_k}{k!} (k-1) z^{k-2}. \tag{13}$$

Exercise 1.4. *Show that*

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$.

The modular properties of Eisenstein series follow from (4). We find that for $k \geq 4$

$$E_k(\gamma.\tau) = (c\tau + d)^k E_k(\tau). \tag{14}$$

Thus $E_k(\tau)$ is a (holomorphic) modular form of weight k for $SL(2, \mathbf{Z})$. Every modular form can be expressed as a polynomial in E_4 and E_6 [Se].

Example 1.5. $E_8 = \frac{3}{7} E_4^2$ *and* $E_{10} = \frac{5}{11} E_4 E_6$.

The series $E_2(\tau)$ is called a quasimodular form having the exceptional modular transformation

$$E_2(\gamma.\tau) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}, \tag{15}$$

which follows from a deeper analysis of the order of the double sum [Se]. Every quasimodular form can be expressed as a polynomial in E_2, E_4 and E_6

One important application of E_2 is the following: Let f_n be a modular form of weight n. We define the *modular derivative* of f_n by

$$D f_n = (q \frac{\partial}{\partial q} + n E_2) f_n. \tag{16}$$

Lemma 1.6. *Df_n is a modular form of weight $n+2$.*

Exercise 1.7. *Prove (16).*

Example 1.8. *$DE_4 = 14E_6$ and $DE_6 = \frac{60}{7}E_4^2$.*

Exercise 1.9. *Show for $f_{12} = 20E_4^3 - 49E_6^2$ that $Df_{12} = 0$. Show that $f_{12} = O(q)$ i.e. f_{12} is a cusp form of weight 12. In fact $f_{12} = \frac{1}{2^4 3^3 5^2}\Delta$ where $\Delta = \eta(q)^{24} = q\prod_{n\geq 1}(1-q^n)^{24}$. Thus it also follows that $D\eta = 0$.*

Finally, by definition P_1 is periodic with period $2\pi i$. However, P_1 is not an elliptic function since

$$P_1(z + 2\pi i\tau, \tau) = P_1(z,\tau) - 1,$$

(which follows from (15)). Nevertheless, all the z derivatives $P_1^{(m)}(z,\tau) = \frac{d^m}{dz^m}P_1(z,\tau)$ for $m \geq 1$ are elliptic functions given by

$$\begin{aligned} P_1^{(m)}(z,\tau) &= \sum_{n\in\mathbf{Z}}{}' \frac{n^m q_z^n}{1-q^n} \\ &= m!\left[\frac{(-1)^{m+1}}{z^{m+1}} + \sum_{k\geq m+1}\binom{k-1}{m}E_k(\tau)z^{k-m-1}\right]. \end{aligned} \tag{17}$$

2 Modular and Elliptic Functions for Vertex Operator Algebras

2.1 Review of Vertex Operator Algebras

We review some aspects of Vertex Operator Algebra theory to establish context and notation. For more details see [FHL], [FLM], [Ka], [MN]. A Vertex Operator Algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ as follows:

- V is a vector space with a non-negative $\mathbf{Z}$-grading where

  $$V = \oplus_{n\geq 0}V_n.$$

  $\mathbf{1} \in V_0$ is the vacuum vector and $\omega \in V_2$ the conformal vector with properties described below.

- Y is a linear map $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$, for formal variable z, so that for any vector $u \in V$

$$Y(u,z) = \sum_{n \in \mathbf{Z}} u(n) z^{-n-1}.$$

  The modes $u(n) \in \mathrm{End}V$ are such that $u(n)\mathbf{1} = \delta_{n,-1} u$ for $n \geq -1$.

- Vertex operators satisfy the locality property for all $u, v \in V$

$$(x-y)^N [Y(u,x), Y(v,y)] = 0, \tag{18}$$

  for $N \gg 0$.

- The vertex operator for the vacuum is $Y(\mathbf{1}, z) = Id_V$ whereas that for ω is

$$Y(\omega, z) = \sum_{n \in \mathbf{Z}} L(n) z^{-n-2}, \tag{19}$$

  where $L(n) = \omega(n+1)$ forms a Virasoro algebra for central charge c

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{20}$$

  $L(-1)$ satisfies the translation property

$$Y(L(-1)u, z) = \frac{d}{dz} Y(u,z). \tag{21}$$

  $L(0)$ describes the $\mathbf{Z}$-grading with $L(0)u = wt(u)u$ for weight $wt(u) \in \mathbf{Z}$ and

$$V_n = \{u \in V | wt(u) = n\}. \tag{22}$$

We quote some standard properties of VOAs following from these axioms e.g. [Ka], [FHL], [MN]. We have a commutator identity

$$[u(k), Y(v,z)] = \sum_{i \geq 0} \binom{k}{i} Y(u(i)v, z) z^{k-i}. \tag{23}$$

Taking $u = \omega$ this implies for v of weight $wt(v)$ that

$$[L(0), v(m)] = (wt(v) - m - 1)v(m), \tag{24}$$

so that

$$v(m)V_n \subset V_{n+wt(v)-m-1}.$$

In particular, we define for v of weight $wt(v)$ the zero mode

$$o(v) = v(wt(v) - 1), \tag{25}$$

which is then extended by linearity to all $v \in V$. Then

$$o(v)V_n \subset V_n. \tag{26}$$

Exercise 2.1. *Show that*

$$q_x^{L(0)} Y(v, z) q_x^{-L(0)} = Y(q_x^{L(0)} v, q_x z), \tag{27}$$

where $q_x^{L(0)} = \exp(xL(0))$.

Exercise 2.2. *Show that*

$$o((L(-1) + L(0))v) = 0. \tag{28}$$

v is said to be a primary vector if

$$L(n)v = 0, \tag{29}$$

for all $n > 0$. A primary vector is a highest weight vector for a Virasoro Verma module.

Exercise 2.3. *Show that* v *is primary iff*

$$L(1)v = L(2)v = 0. \tag{30}$$

2.2 The Square Bracket Formalism

In order to consider modular-invariance of n-point functions at genus 1, Zhu introduced in ref. [Z] an isomorphic "square-bracket" VOA $(V, Y[,], \mathbf{1}, \tilde{\omega})$ associated to a given VOA $(V, Y(,), \mathbf{1}, \omega)$. The main purpose is to construct vertex operators that are automatically periodic in $2\pi i$ and hence "live" on the cylinder $z \sim z + 2\pi i$

The new square bracket vertex operators are defined by

$$Y[v, z] = \sum_{n \in \mathbf{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1), \tag{31}$$

with $q_z = \exp(z)$, while the new conformal vector is

$$\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}. \tag{32}$$

For v of $L(0)$ weight $wt(v)$ and $m \geq 0$ then

$$v[m] = m!\sum_{i\geq m} c(wt(v), i, m)v(i), \tag{33}$$

$$\sum_{m=0}^{i} c(wt(v), i, m)x^m = \binom{wt(v) - 1 + x}{i}. \tag{34}$$

From (33) and (34) we find

$$\sum_{i\geq 0} \binom{k}{i} v(i) = \sum_{m\geq 0} \frac{(k + 1 - wt(v))^m}{m!} v[m]. \tag{35}$$

These identities are proved in the Appendix.

Exercise 2.4. *Show that* $L[-1] = L(-1) + L(0)$.

The Virasoro operator $L[0]$ provides us with an alternative $\mathbf{Z}$-grading structure on V where $L[0]u = wt[u]u$ for square bracket weight $wt[u] \in \mathbf{Z}$. Then

$$\begin{aligned} V &= \oplus_{n\geq 0} V_{[n]}, \\ V_{[n]} &= \{u \in V | wt[u] = n\}. \end{aligned}$$

Furthermore, v is primary with respect to the "round bracket" Virasoro algebra $\{L(n)\}$ iff v is primary with respect to $\{L[n]\}$. In addition, $wt(v) = wt[v]$.

Example 2.5. *The rank 1 Heisenberg VOA M is generated by a primary vector a with $wt(a) = 1$ with V a Fock space spanned by vectors of the form $a(-k_1)\ldots a(-k_n)\mathbf{1}$ for $1 \leq k_1 \ldots \leq k_n$.Alternatively, in the square bracket formalism we can choose a basis of Fock vectors of the form $a[-k_1]^{e_1}\ldots a[-k_n]\mathbf{1}$ where (as before)*

$$[a[m], a[n]] = m\delta_{m+n,0}.$$

2.3 Torus 1-point and 2-point Correlation Functions

We now describe the relationship between VOAs and elliptic and modular functions in terms of torus 1-point and 2-point functions. We recall the VOA *partition function*

$$Z_V(\tau) = \mathrm{Tr}_V\left(q^{L(0)-c/24}\right) = \sum_{n\geq 0} \dim V_n q^{n-c/24}. \tag{36}$$

The central charge factor is included to enhance the modular properties of $Z(\tau)$. Thus for the rank 1 Heisenberg VOA M we find

$$Z_M(\tau) = q^{-1/24}\prod_{n\geq 1}(1-q^n)^{-1} = \frac{1}{\eta(\tau)},$$

is an $SL(2,\mathbf{Z})$ meromorphic modular form of weight $-1/2$ with a multiplier system. For a Heisenberg module M_β with $a(0)$ eigenvalue β the partition function is

$$Z_{M_\beta}(\tau) = \frac{q^{\beta^2/2}}{\eta(\tau)}.$$

This implies that for a rank l lattice VOA V_L the partition function is

$$Z_{V_L}(\tau) = \frac{\theta_L}{\eta^l},$$

for lattice theta function

$$\theta_L(\tau) = \sum_{\alpha\in L} q^{\langle\alpha,\alpha\rangle/2}.$$

$Z_{V_L}(\tau)$ is modular invariant under a subgroup of $SL(2,\mathbf{Z})$.

We define the *1-point correlation function* for $u \in V$ by

$$\begin{aligned} Z_V(u,\tau) &= \mathrm{Tr}_V(Y(q_z^{L(0)}u, q_z)q^{L(0)-c/24}) \\ &= \mathrm{Tr}_V(o(u)q^{L(0)-c/24}), \end{aligned} \tag{37}$$

where $o(u)$ is the zero mode (25). Notice that $Z_V(u,\tau)$ is inde) where $q_i = q_{z_i}$. The 2-point function can also be expressed in terms of a 1-point function by using associativity of VOAs ([FHL]) and scaling (27) so that the RHS of (38) is

$$\begin{aligned} & F_V((u,z_1),(v,z_2),\tau) = \mathrm{Tr}_V(Y(Y(q_1^{L(0)}u, q_1 - q_2)q_2^{L(0)}v, q_2)q^{L(0)-c/24}) \\ = & \ \mathrm{Tr}_V(Y(q_2^{L(0)}Y((\frac{q_1}{q_2})^{L(0)}u, \frac{q_1}{q_2}-1)v, q_2)q^{L(0)-c/24}) \\ = & \ Z_V(Y[u,z_{12}]v,\tau), \end{aligned} \tag{39}$$

which is a function of $z_{12} = z_1 - z_2$. F_V is also clearly periodic in z_i with period $2\pi i$. Furthermore

$$\begin{aligned} F_V((u,z_1),(v,z_2+2\pi i\tau),\tau) &= q^{-c/24}\mathrm{Tr}_V(Y(q_1^{L(0)}u,q_1)Y(q^{L(0)}q_2^{L(0)}v,qq_2)q^{L(0)}) \\ &= q^{-c/24}\mathrm{Tr}_V(Y(q_1^{L(0)}u,q_1)q^{L(0)}Y(q_2^{L(0)}v,q_2)) \\ &= q^{-c/24}\mathrm{Tr}_V(Y(q_2^{L(0)}v,q_2)Y(q_1^{L(0)}u,q_1)q^{L(0)}) \\ &= F_V((u,z_1),(v,z_2),\tau), \end{aligned}$$

from locality (18). Thus F_V is also periodic in z_i with period $2\pi i\tau$. Hence *provided* F_V is analytic then it is elliptic in z_i.

2.4 Zhu Recursion Formulas

We now show that F_V is indeed elliptic and is given by a recursion formula due to Zhu [Z]

Theorem 2.6.

$$\begin{aligned} F_V((u,z_1),(v,z_2),\tau) &= \mathrm{Tr}_V\left(o(u)o(v)q^{L(0)-c/24}\right) \\ &\quad + \sum_{m\geq 1}\frac{(-1)^{m+1}}{m!}P_1^{(m)}(z_{12},\tau)Z_V(u[m]v,\tau). \end{aligned} \tag{40}$$

Remark 2.7. *The sum in (40) is finite since $u[m]v = 0$ for m sufficiently large and since each coefficient function $P_1^{(m)}(z_{12},\tau)$ is elliptic, the two point function is elliptic.*

To prove Theorem 2.6 we first assume wlog that u is of weight $wt(u)$ so that

$$F_V((u,z_1),(v,z_2),\tau) = \sum_{k\in\mathbf{Z}} q_1^{-k-1+wt(u)}\mathrm{Tr}_V\left(u(k)Y(q_2^{L(0)}v,q_2)q^{L(0)-c/24}\right). \tag{41}$$

Applying the locality commutator formula (23), scaling (27) and (35) we find

$$\begin{aligned}\left[u(k), Y(q_2^{L(0)}v, q_2)\right] &= \sum_{i\geq 0}\binom{k}{i} Y(u(i)q_2^{L(0)}v, q_2)q_2^{k-i} \\ &= q_2^{k+1-wt(u)} Y(q_2^{L(0)}\sum_{i\geq 0}\binom{k}{i}u(i)v, q_2) \\ &= q_2^n \sum_{m\geq 0}\frac{n^m}{m!}Y(q_2^{L(0)}u[m]v, q_2),\end{aligned}$$

where, for convenience, we have defined

$$n = k+1-wt(u) \in \mathbf{Z}.$$

Hence we find

$$\begin{aligned}&\mathrm{Tr}_V\left(u(k)Y(q_2^{L(0)}v, q_2)q^{L(0)-c/24}\right) \\ =\ &\mathrm{Tr}_V\left([u(k), Y(q_2^{L(0)}v, q_2)]q^{L(0)-c/24}\right) + \mathrm{Tr}_V\left(Y(q_2^{L(0)}v, q_2)u(k)q^{L(0)-c/24}\right) \\ =\ &q_2^n\sum_{m\geq 0}\frac{n^m}{m!}Z_V(u[m]v,\tau) + q^n\mathrm{Tr}_V\left(Y(q_2^{L(0)}v, q_2)q^{L(0)-c/24}u(k)\right),\end{aligned}$$

using (27) again. Finally applying the standard trace identity $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ we find

$$(1-q^n)\mathrm{Tr}_V\left(u(k)Y(q_2^{L(0)}v, q_2)q^{L(0)-c/24}\right) = q_2^n\sum_{m\geq 0}\frac{n^m}{m!}Z_V(u[m]v,\tau). \quad (42)$$

For $n=0$ this implies

$$Z_V(u[0]v,\tau) = 0, \quad (43)$$

whereas for $n\neq 0$ we find

$$\mathrm{Tr}_V\left(u(k)Y(q_2^{L(0)}v, q_2)q^{L(0)-c/24}\right) = \frac{q_2^n}{1-q^n}\sum_{m\geq 1}\frac{n^m}{m!}Z_V(u[m]v,\tau).$$

Hence substituting into (41) and recalling (25) we find

$$\begin{aligned}F_V((u,z_1),(v,z_2),\tau) &= \mathrm{Tr}_V\left(o(u)o(v)q^{L(0)-c/24}\right) \\ &\quad + \sum_{m\geq 1}\frac{1}{m!}Z_V(u[m]v,\tau)\sum_{n\in\mathbf{Z}}{}^{'}\frac{n^m(\frac{q_2}{q_1})^n}{1-q^n}.\end{aligned}$$

Comparing to (17) the result follows. ■

Exercise 2.8. *For the Heisenberg VOA generated by a show that*

$$F_M((a, z_1), (a, z_2), \tau) = \frac{P_2(z_{12}, \tau)}{\eta(\tau)}.$$

Theorem 2.6 allows us to obtain a related recursion formula for 1-point functions.

Proposition 2.9. *For $k \geq 1$*

$$\begin{aligned} Z_V(u[-k]v, \tau) &= \delta_{k,1} \mathrm{Tr}_V(o(u)o(v)q^{L(0)-c/24}) \\ &\quad + \sum_{m \geq 1} (-1)^{m+1} \binom{k+m-1}{m} E_{k+m}(\tau) Z_V(u[m]v, \tau). \end{aligned} \tag{44}$$

To prove this firstly note from (39) that

$$F_V((u, z_1), (v, z_2), \tau) = \sum_{k \in \mathbf{Z}} Z_V(u[-k]v, \tau) z_{12}^{-k-1}.$$

We may compare this to the z_{12} expansion of the RHS of (40) using the expansion of $P_1^{(m)}(z_{12}, \tau)$ in (17). For $k \leq 0$ this amounts to trivial equality between singular terms. For $k \geq 1$ we obtain the result. ■

3 Examples of 1-Point Functions

3.1 The Heisenberg VOA

Consider the Heisenberg VOA M and let $u = a[-k_1] \ldots a[-k_n]\mathbf{1}$ for $1 \leq k_1 \ldots \leq k_n$.be an arbitrary Fock vector in the square bracket formalism with $L[0]$ weight $wt[u] = \sum_{i=1..n} k_i$. $Z_M(u, \tau)$ can be computed by iterating (44) of Proposition 2.9 as follows

$$Z_M(a[-k_1]a[-k_2] \ldots a[-k_n]\mathbf{1}, \tau) =$$
$$0 + \sum_{i=2}^{n} (-1)^{k_i+1} k_i \binom{k_1+k_i-1}{k_i} E_{k_1+k_i}(\tau) Z_M(a[-k_2] \ldots \hat{a}[-k_i] \ldots a[-k_n]\mathbf{1}, \tau),$$

where $\hat{a}[-k_i]$ denotes that the operator $a[-k_i]$ is deleted. Noting that $E_{k_1+k_i}(\tau)$ is a quasi-modular form of weight $k_1 + k_i \geq 2$ one then finds

Proposition 3.1. *For u of $L[0]$ weight $wt[u]$ then*

$$Z_M(u,\tau) = \frac{Q(\tau)}{\eta(\tau)},$$

where $Q(\tau)$ is a quasi–modular form of weight $wt[u]$.

Example 3.2. $Z_V(a[-1]^2\mathbf{1},\tau) = \frac{E_2(\tau)}{\eta(\tau)}$.

Exercise 3.3. *Show that $Q(\tau)$ is a modular form of weight $wt[u]$ iff $k_2 \geq 2$.*

These results can be easily extended to include 1-point functions for a Heisenberg module M_β to find

$$\begin{aligned} & Z_{M_\beta}(a[-k_1]a[-k_2]\ldots a[-k_n]\mathbf{1},\tau) = \\ & \delta_{k_1,1}\beta Z_{M_\beta}(a[-k_2]\ldots a[-k_n]\mathbf{1},\tau) \\ + \sum_{i=2}^{n}(-1)^{k_i+1}k_i\binom{k_1+k_i-1}{k_i} & E_{k_1+k_i}(\tau)Z_{M_\beta}(a[-k_2]\ldots \hat{a}[-k_i]\ldots a[-k_n]\mathbf{1},\tau). \end{aligned} \tag{45}$$

3.2 Lattice VOAs

Consider a rank 1 lattice VOA V_L with Fock vector $u = a[-k_1]\ldots a[-k_n]\mathbf{1}$ of weight $wt[u]$ for $1 \leq k_1 \ldots \leq k_n$. Then

Proposition 3.4. *$Z_{V_L}(u,\tau)$ is modular form of weight $wt[u]$ for the subgroup of $SL(2,\mathbf{Z})$ which preserves $Z_{V_L}(\tau) = \theta_L/\eta$.*

We can firstly prove this for a square bracket Virasoro primary u using

Lemma 3.5. *If u is a primary vector for the square bracket Virasoro then $k_2 \geq 2$.*

Exercise 3.6. *Prove by showing that $L[2]a[-1]^2\ldots a[-k_n]\mathbf{1} \neq 0$.*

Hence we may apply (45) to prove the Proposition in this case.

The general result follows by considering Virasoro descendents of a primary. Consider the square bracket Virasoro $\tilde{\omega}$ with $L[k] = \tilde{\omega}[k+1]$. Then (44) implies

$$\begin{aligned} Z_V(L[-k]v,\tau) &= \delta_{k,2}\mathrm{Tr}_V(o(\tilde{\omega})o(v)q^{L(0)-c/24}) \\ &\quad + \sum_{m\geq 0}(-1)^m\binom{k+m-1}{m+1}E_{k+m}(\tau)Z_V(L[m]v,\tau). \end{aligned} \tag{46}$$

But $o(\tilde{\omega}) = L(0) - c/24$ and hence

$$\mathrm{Tr}_V(o(\tilde{\omega})o(v)q^{L(0)-c/24}) = q\frac{\partial}{\partial q}Z_V(v,\tau).$$

It follows that for v of $L[0]$ weight $wt[v]$

$$Z_V(L[-2]v,\tau) = \left(q\frac{\partial}{\partial q} + wt[v]E_2(\tau)\right)Z_V(v,\tau) + \sum_{m\geq 2, even} E_{2+m}(\tau)Z_V(L[m]v,\tau). \tag{47}$$

We can thus apply (46) and (47) iteratively to compute the 1-point function for every Virasoro descendent $L[-k_1]\ldots L[-k_n]u$ of a primary u to prove Proposition 3.4. In particular, we find that (47) then reads

$$Z_V(L[-2]v,\tau) = DZ_V(v,\tau) + \sum_{m\geq 2, even} E_{2+m}(\tau)Z_V(L[m]v,\tau),$$

where D is the modular derivative of (16). ■

Proposition 3.4 illustrates a more general theorem that holds for all rational and C_2 cofinite VOAs. The above techniques also play a major role in proving convergence and modular invariance of the partition functions for V and its modules. Thus one finds for such theories that the 1-point function for some Virasoro descendent of a primary u vanishes leading to a modular invariant differential equation satisfied by $Z_V(u,\tau)$.

4 Appendix A: The Square Bracket Formalism

We prove (33)-(35). The square bracket vertex operator v of $L(0)$ weight $wt(v)$ is defined by

$$Y[v,z] = q_z^{wt(v)}Y(v,q_z-1).$$

Thus the square bracket modes of $Y[v,z] = \sum_{m\in\mathbf{Z}} v[m]z^{-m-1}$ are given by

$$\begin{aligned} v[m] &= \text{coeff of } z^{-1} \text{ in } Y(v,q_z-1)z^m q_z^{wt(v)} \\ &= \text{coeff of } z^{-1} \text{ in } Y(v,q_z-1)\frac{d}{dz}(q_z-1)z^m q_z^{wt(v)-1}. \end{aligned}$$

We may rewrite this in terms of $w = q_z - 1 = z + O(z^2)$ by means of a (formal) chain rule [FHL], [Z] so that

$$\begin{aligned} v[m] &= \text{coeff of } w^{-1} \text{ in } Y(v,w)z(w)^m q_{z(w)}^{wt(v)-1} \\ &= \text{coeff of } w^{-1} \text{ in } Y(v,w)\ln(1+w)^m(1+w)^{wt(v)-1}. \end{aligned}$$

Define $c(wt(v),i,m)$ for $i \geq m \geq 0$ by

$$\sum_{i\geq m} c(wt(v),i,m)w^i = \frac{1}{m!}\ln(1+w)^m(1+w)^{wt(v)-1},$$

we thus find (33). Next note that $\sum\limits_{m\geq 0}\frac{1}{m!}\ln(1+w)^m x^m = (1+w)^x$. Hence we find

$$\sum_{i\geq 0}\sum_{m=0}^{i} c(wt(v),i,m)w^i x^m = (1+w)^{wt(v)-1+x},$$

from which (34) follows. Finally considering

$$\begin{aligned} \sum_{m\geq 0}\frac{(k+1-wt(v))^m}{m!}v[m] &= \sum_{m\geq 0}\sum_{i\geq m} c(wt(v),i,m)(k+1-wt(v))^m v(i) \\ &= \sum_{i\geq 0} v(i)\sum_{m=0}^{i} c(wt(v),i,m)(k+1-wt(v))^m, \\ &= \sum_{i\geq 0}\binom{k}{i}v(i). \end{aligned}$$

giving (35).

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