INTRODUCTION TO SYMPLECTIC GEOMETRY AND TOPOLOGY

Exercises for Lecture 2

Exercise 9. *(Do enough to get a good feel for the relationships between symplectic structures, complex structures and inner products.)* Prove that the following are equivalent for any complex structure J on a symplectic vector space (V, ω).

(1) J is ω-compatible.
(2) The bilinear form $g_J(u,v) \coloneqq \omega(u, Jv)$ defines a J-invariant inner product.
(3) The bilinear form $h_J(u,v) \coloneqq \omega(u, Jv) + i\omega(u,v)$ is Hermitian,

$$h_J(Ju, v) = -ih_J(u,v) \qquad h_J(v,u) = \overline{h_J(u,v)}$$
$$h_J(u,v) = ih_J(u,v) \qquad \text{and}$$

Re h_J is symmetric and positive definite.

Exercise 10. Prove that a complex structure J is ω-compatible iff there exists a standard symplectic basis $\{u_i, v_i\}$ such that $v_i = Ju_i$ for each i

Exercise 11. Prove that if $A \in \mathrm{Sp}(2n)$ then $(AA^T)^\alpha \in \mathrm{Sp}(2n)$ for all $\alpha \in \mathbb{R}$. Hint: Use the fact that AA^T preserves ω and is diagonal with respect to some basis.

Exercise 12. Prove the contractiblity of the space of symplectic structures ω for which a given complex structure J is ω-compatible. Hint: Consider symplectic structures of the form $\omega_B(v,w) = w^T B J_0 v$.

Exercise 13. *(Important because cotangent bundles are the model for all neighborhoods of Lagrangian submanifolds.)* Verify that the following three characterizations of the Liouville form λ on a cotangent bundle $\pi : T^*L \to L$ are equivalent.

(1) For any point $q \in L$, covector $v_q^* \in T_q^*L$ and tangent vector $w \in T_{v_q^*}(\pi_* w)$, we have $\lambda(w) = v_q^*(\pi_* w)$.
(2) For any 1-form σ on L, i.e. any section $\sigma : L \to T^*L$, $\sigma^*\lambda = \sigma$.
(3) If (q,p) are local coordinates on T^*L with q and p being coordinates on the base and fiber respectively, then $\lambda = \sum_i p_i \wedge dq_i$.

Exercise 14. *(Straightforward application of the characterization of the Liouville form.)* Prove that a 1-form σ on a smooth manifold L is closed if and only if the graph of σ is a Lagrangian submanifold of $(T^*L, d\lambda)$.

Two Sided →

Exercise 15. *(The manifold version of Exercise 6.)* A diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ is a symplectomorphism if and only if its graph, $\{(x, \phi(x)) \in (M \times M, (-\omega) \oplus \omega)\}$, is Lagrangian.

Exercise 16. *(This is a key step in Moser's method. Do verify it.)* Suppose $\frac{d}{dt}\omega_t = d\sigma_t$ for all t in some interval I containing 0. Let X_t be the family of vector fields defined by $\sigma_t + \iota_{X_t}\omega_t = 0$, and let ψ_t the flow generated by X_t, so $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ and $\psi_0 = Id$. Verify that $\frac{d}{dt}\psi_t^*\omega_t = 0$. Hint: Use Cartan's formula for the Lie derivative of a form: $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\alpha$.

Exercise 17. *A bit more open ended...* What are the statements and proofs of neighborhood theorems for isotropic and coisotropic submanifolds, in particular, hypersurfaces?