(II) THE BRAUER GROUP OF A VARIETY

II.1. Brauer group of a field

k perfect field (usually of char 0), $\bar{k}$ alg. closure,

$$\Gamma = \mathrm{Gal}(\bar{k}/k) = \varprojlim_{\substack{L/k \text{ finite} \\ \text{Galois}}} \mathrm{Gal}(L/k)$$

A abelian group equipped with a continuous action of Γ := Γ-module.

examples: $\cdot$ $A = \mathbb{Z}/n$ (trivial action)

$\cdot$ $A = \mu_n(\bar{k})$

$\cdot$ $A = G(\bar{k})$, G commutative alg. k-group.

For a Γ-module A, define the Galois cohomology groups $H^i(\Gamma, A) = H^i(k, A)$ for $i \geq 0$.

e.g.: $H^0(\Gamma, A) = A^\Gamma$ ("invariants").

Some properties:

- If the action is trivial, then $H^1(\Gamma, A) = \mathrm{Hom}_c(\Gamma, A)$.
- $H^i(\Gamma, A)$ is covariant in A, contravariant in Γ.
- If $0 \to A \to B \to C \to 0$ is a short exact sequence of Γ-modules, then there is a long exact sequence

$$0 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1(\Gamma, A) \to H^1(\Gamma, B) \to H^1(\Gamma, C) \to H^2(\Gamma, A) \to \cdots$$

<u>examples</u>: . $H^1(\Gamma, \bar{k}^*) = 0$ (Hilbert's 90).

. If $\mathrm{char}\, k \nmid n$, then

$$H^1(\Gamma, \mu_n(\bar{k})) = k^*/k^{*n}$$

(Kummer theory; use the exact sequence $0 \to \mu_n(\bar{k}) \to \bar{k}^* \xrightarrow{\times n} \bar{k}^* \to 1$ and Hilbert's 90).

<u>Rem</u>: For $i \geq 1$, the group $H^i(\Gamma, A)$ is torsion.

$$\left(H^i(\Gamma, A) = \varinjlim H^i(\mathrm{Gal}(L/k), A^{\mathrm{Gal}(\bar{k}/L)}) \right)$$

<u>Def</u>: The Brauer group of k is $H^2(\Gamma, \bar{k}^*) = H^2(k, \mathbb{G}_m)$.

<u>Ex</u>: . $\mathrm{Br}\,\mathbb{C} = 0$, similarly for $\mathbb{F}_q$, $\mathbb{C}(t)$.

. $\mathrm{Br}\,\mathbb{R} = \mathbb{Z}/2$.

. $\mathrm{Br}\,\mathbb{Q}_p = \mathbb{Q}/\mathbb{Z}$ (local class field theory).

. For a number field k, there is an exact sequence:

$$0 \to \mathrm{Br}\,k \to \bigoplus_{v \in \Omega_k} \mathrm{Br}\,k_v \xrightarrow{\Sigma j_v} \mathbb{Q}/\mathbb{Z} \to 0$$

(global c.f.t.). "reciprocity for Br."

Other definition of Br k:

Br k can be defined as equivalence classes of central simple algebras over k, with the law $\otimes$.

$A \sim B$ iff $A \simeq M_n(D)$, $B \simeq M_m(D)$ for the same division algebra D.

Thus $\mathrm{Br}\, k = 0 \iff$ every division algebra with center k is $\simeq k$.

(char $k \neq 2$)

Ex: • Let a, b in k^*. Then the Hilbert symbol (a,b) is an element of $\mathrm{Br}\,k\,[2]$. It corresponds to the quaternion algebra: k-algebra with basis $(1, i, j, k)$, relations $i^2 = -ab$, $j^2 = a$, $k^2 = b$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

For $k = \mathbb{R}$ or $\mathbb{Q}_p$, $\mathrm{Br}\,k\,[2] = \mathbb{Z}/2$, recover the def. in the 1. talk.

• (a,b) can be defined as a cup-product in group cohomolog.

II.2. Generalization to arbitrary schemes

Let X be an algebraic variety over a field k, G a commutative alg. k-group (ex. $G = \mathbb{Z}/n$, $G = \mu_n$, $G = \mathbb{G}_m$).

Then one can define étale cohomology groups $H^i(X, G)$ for $i \geqslant 0$.

If $X = \operatorname{Spec} k$, then $H^i(X, G) = H^i(k, G(\bar{k}))$ (Galois cohomology group).

<u>Properties</u>: $\cdot$ $H^0(X, G) = G(X) = \mathrm{Mor}_k(X, G)$

$\cdot$ $H^i(X, G)$ is covariant in G, contravariant in X.

$\cdot$ long exact sequence associated to an exact sequence $0 \to G_1 \to G_2 \to G_3 \to \cdot$ of k-groups.

<u>Examples</u>: $\cdot$ $H^1(X, G_m) = \mathrm{Pic}\, X$

$\cdot$ If X is projective over an alg. closed field k, then

$H^1(X, \mu_n) = \mathrm{Pic}\, X\,[n]$ $\quad (n, \mathrm{char}\, k) = 1$

(use $0 \to \mu_n \to G_m \xrightarrow{n} G_m \to 0$).

<u>Def</u>: The (cohomological) Brauer group of X is
$H^2(X, \mathbb{G}_m) = \mathrm{Br}\, X$

<u>ex</u>: $\mathrm{Br}(\mathrm{Spec}\, k) = \mathrm{Br}\, k$.

(can be defined similarly for any scheme X).

<u>Th</u> [Grothendieck] For X <u>smooth</u> over k and integral,

$$\mathrm{Br}\, X \hookrightarrow \mathrm{Br}(\underbrace{k(X)}_{\text{function field of } X.})$$

Hence $\mathrm{Br}\, X$ is torsion.

II.3. The Brauer-Manin obstruction

X projective and smooth variety over a number field k.

$X(\mathbb{A}_k) := \prod_{v \in \Omega_k} X(k_v)$ set of adèlic points.

Assume $X(\mathbb{A}_k) \neq \emptyset$ Local conditions.

Let $\alpha \in \mathrm{Br}\, X$. For $P_v \in X(k_v)$, evaluate $\alpha(P_v) \in \mathrm{Br}\, k_v$, then take the local invariant $j_v(\alpha(P_v)) \in \mathbb{Q}/\mathbb{Z}$.

Then, by reciprocity law for $Br\, k$, if $(P_v)_{v \in \Omega_k}$ comes from a rational point $P \in X(k)$, then:

$$\boxed{\sum_{v \in \Omega_k} j_v(\alpha(P_v)) = 0} \quad \text{for any } \alpha \in Br\, X$$

"Manin conditions" (Manin, ICM 70)

ex: Iskovskih: X smooth and proj. model of $y^2 + z^2 = (x^2-2)(3-x^2)$

$\alpha = (-1, x^2-2) \in Br(k(X))$ is in fact in $Br\, X$ and for any $(P_v) \in X(\mathbb{A}_k)$, $\sum_v j_v(\alpha(P_v)) \neq 0$

$\Rightarrow$ Manin obstruction to the Hasse principle

Similarly, if $(P_v) \in X(\mathbb{A}_k)$ is such that $\sum_v j_v(\alpha(P_v)) \neq 0$ for some $\alpha \in \mathrm{Br}\, X$, then (P_v) is not in the closure of $X(k)$ for the product of v-adic topologies

→ Brauer-Manin obstruction to weak approximation. (Colliot-Thélène / Sansuc).

<u>Rem</u>: . "Constant" elements of Br X do not give any obstruction.

. If $X \times_k \bar{k}$ is rational, $\mathrm{Br}\, X / \mathrm{Br}\, k$ is finite → BM obs. is "computable"

II.4. Some results and conjectures

Conjecture [Colliot-Thélène / Sansuc]

For a (geometrically) rational surface X, Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one.

known for:
- Intersection of two quadrics in $\mathbb{P}^n$, $n \geq 8$. (CT/Sansuc, Swinnerton-Dyer, 1987).
- Diagonal cubic surfaces over $\mathbb{Q}$, assuming finiteness of Ш (Swinnerton-Dyer, 2000). + some additional condition.

Th [Sansuc, 1981]: The analogue holds for principal homogeneous spaces of linear algebraic groups.

Th [Manin, 1970; Wang 1996]: likewise for abelian variety (generalization to any connected algebraic group, D.H. 2004).

Th [Skorobogatov 1997] There exists a bielliptic surface $X/\mathbb{Q}$ such that:

i) $X(\mathbb{Q}) = \emptyset$

ii) There exists an adelic point on X satisfying Manin conditions!