Arithmetic of linear algebraic groups and their homogeneous spaces

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k number field

1) HP holds for PHS under semi-simple simply connected linear algebraic groups

2) HP holds for projective homogeneous varieties under connected linear algebraic groups

3) A_G - obstruction to weak approximation is finite

4) $Ш^1_G$ - obstruction to Hasse principle is finite

3), 4) for any connected linear algebraic group

and
5) $G(k)/R$ finite abelian for any connected linear algebraic group G/k

"Dimension 2" fields

Conjecture II of Serre - vast generalisation of Kneser's Conjecture:

A field k is said to have cohomological dimension at most n ($cd(k) \leq n$) if for every finite discrete Γ_k-module M, $H^r(\Gamma_k, M) = 0$ for $r \geq n+1$

e.g. $\mathbb{F}_q$, $\mathbb{C}(t)$ cohomological dimension 1

$\mathbb{Q}_p$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{C}(t_1, t_2)$ cohomological dimension 2

Conjecture II (Serre) Let k be a perfect field, with $cd(k) \leq 2$. Let G be a semi-simple simply connected linear algebraic group defined over k. Then $H^1(k, G) = \{1\}$

Arithmetic case: p-adic fields: Kneser

Number fields without ordering: consequence of Kneser Conjecture: Kneser, Harder, Chernousov.

The first major breakthrough for a general ground field of cohomological dimension 2 is due to <u>Merkurjev</u> and <u>Suslin</u>:

<u>Theorem</u> Let k be a perfect field with $cd(k) \le 2$. Let D be a central division algebra over k. Then

$H^1(k, SL_1(D)) = \{1\}$

More:
The following are equivalent for a perfect field k:

1) $cd(k) \le 2$

2) $\forall$ finite extension ℓ/k and every central division algebra D/ℓ, $Nrd : D^* \longrightarrow \ell^*$ is surjective

Conjecture II was settled in the affirmative for other classical groups by Eva Bayer-Fluckiger, —

Proof uses Merkurjev-Suslin theorem.

Classical groups

An absolutely simple simply connected group G is said to be of classical type if G^{ad} is the component of identity of a group of automorphisms of a central simple algebra with an involution.

Weil: Groups of classical type are precisely of the type A_n, B_n, C_n, D_n (D_4 non-trialitarian)

We give a list of these groups:

A central simple algebra $/K$

$\sigma : A \longrightarrow A$ an involution

(eg. $M_n(K) \longrightarrow M_n(K)$, $X \longrightarrow X^t$)

σ is unitary if $\sigma/K \neq$ identity ; $K^\sigma = k$.

Suppose $\sigma/K =$ identity ; $K = k$.

σ is orthogonal if $\sigma_{\bar{k}} : X \longrightarrow X^t$

σ is symplectic if $\sigma_{\bar{k}} : X \longrightarrow E X^t E^{-1}$,

$$E = \begin{pmatrix} 0, & 1_n \\ -1_n, & 0 \end{pmatrix}$$

^1A_n : $SL_{1,A}$, A central simple algebra $/k$.

^2A_n : $SU(A,\sigma)$, A central simple algebra $/K$ with a unitary K/k involution.

B_n, D_n : $\mathrm{Spin}(A,\sigma)$, A central simple algebra over k with an orthogonal involution

C_n : $SU(A,\sigma)$, A central simple algebra over k with a symplectic involution

(Bayer, —) Conjecture II holds for all groups of classical type and for groups of type G_2 or F_4.

Conjecture II open in general for other exceptional groups - trialitarian D_4, E_6, E_7, E_8.

Proof for classical groups is via classification of hermitian forms over division algebras with involution in terms of invariants: dimension, discriminant, Clifford invariant...

Case F_4:

$H^1(k, G) \simeq$ set of isomorphism classes of exceptional central simple Jordan algebras of dimension 27

Results of Springer lead to Conj. II

There is an analogous conjecture for fields of 'virtual cohomological dimension 2', due to Colliot-Thélène: settled for classical groups by E. Bayer-Fluckiger, ——.

Special fields of cohomological dimension 2

Controlling the kernel $H^1(k, G) \to H^1(l, G)$,

l/k finite cyclic extension of prime degree,

working close to the arithmetic case, led to

the following:

(P. Gille) Let k satisfy

1) $cd(k) \leq 2$
2) index = exponent for central simple algebras over L, L/k finite.

Conjecture II holds for G with no factor of type E_8. Suppose further that $cd(k^{ab}) \leq 1$, if G has a factor of E_8. Then Conjecture II holds for G.

Condition 2) is not satisfied by a general cd 2 field: examples due to Merkurjev.

I $k = \bar{k}$ $\quad \mathrm{char}(k) = 0$

$K = k(X)$; X surface $/k$.

$cd(K) \leq 2$; in fact K is 'C_2'.

(de Jong) index = exponent for central simple algebras over K.

Question. Is $cd(K^{ab}) \leq 1$?

Corollary K function field of a surface over an algebraically closed field. ~~Conjec~~ Then $H^1(K, G) = \{1\}$ for all ss simply connected groups not containing a factor of E_8.

II 2-dimensional strict henselian fields.

A - 2-dimensional excellent henselian local domain with residue field k algebraically closed, characteristic zero.

$K = ff(A) =$ "2-dimensional strict henselian field"

$cd(K) = 2$

eg. 1) $A = k[[X,Y]]$, $K = ff(A) = k((X,Y))$

2) $L / k((X,Y))$ finite extension, B integral closure of A in L.

(Artin), (Ford-Saltman) (Colliot-Thélène - Ojanguren - ——)

index = exponent for central simple algebras $/K$

(all division algebras are cyclic)

Main ingredient in the proof:

Ω_K = set of all rank one discrete valuations of K

$v \in \Omega_K$, $\xi \in Br(K)$, ξ is unramified at v

if $\xi \in$ Image $(Br(\mathcal{O}_v) \longrightarrow Br(K))$

K 2-dimensional strict henselian field.

$$Br_{nr}(K) = \{\xi \in Br(K), \ \xi \text{ unramified at each } v \in \Omega_K\}$$

$$= 0$$

Thus 'splitting the algebra $\Longleftrightarrow$ splitting ramification'

(CT-Gijan-guran, —)

$$cd(K^{ab}) \leq 1$$

<u>Corollary</u> Conjecture II holds for 2-dimensional strict henselian fields.

Study of arithmetic properties like weak approximation, R-equivalence, Hasse principle for homogeneous spaces under connected linear algebraic groups over these 'special' fields was undertaken by Colliot-Thélène, Gille, — and Borovoi-Kunyavskii.

(Colliot-Thélène, Gille, —) Let K satisfy

1) $cd(K) \leq 2$
2) index = exponent for csa $/K$

Let G be a semi-simple simply connected linear algebraic group $/K$, μ = centre of G, If G contains a factor of E_8, suppose $cd(K^{ab}) \leq 1$. Then the connecting map $\delta : H^1(K, G^{ad}) \longrightarrow H^2(K, \mu)$ for the central isogeny

$$1 \longrightarrow \mu \longrightarrow G \longrightarrow G^{ad} \longrightarrow 1$$

is a bijection.

(Analogue of Sansuc's theorem for # fields)

The above result together with Borovoi's techniques in the arithmetic case leads to the following:

(Colliot-Thélène, Gille, —)
Let X be a homogeneous projective variety under a (semi-simple simply) connected group ~~defined~~ over a strict henselian field K. Let Ω_K denote the set of rank one discrete valuations of K. Suppose $X(k_v) \neq \phi$ $\forall v \in \Omega_K$. Then $X(k) \neq \phi$.

(analogue of Harder's theorem for # fields)

Results on R-equivalence, weak approximation, ... for groups defined over these special fields, due to Colliot-Thélène, Gille, Parimala and Borovoi, Kunyavski, can be neatly tied up by a flasque resolution of a connected reductive group, constructed by Colliot-Thélène.

A connected reductive group H is quasi-trivial if it fits into an exact sequence

$$1 \longrightarrow H_1 \longrightarrow H \longrightarrow P \longrightarrow 1$$

with H_1 simply connected, P quasi-trivial torus.

- If $H^1(k, H_1) = \{1\} \Rightarrow H^1(k, H) = \{1\}$
- (Gille) $cd(k) \leq 2$ and $H_1(k)/R = \{1\} \Rightarrow H(k)/R = \{1\}$

Let G be a connected reductive group / k.

A <u>flasque</u> resolution of G is an exact sequence

$$1 \longrightarrow S \longrightarrow H \longrightarrow G \longrightarrow 1$$

with H quasi-trivial and S flasque torus, central in H.

(<u>Colliot</u> - <u>Thélène</u>) Every connected reductive group G admits a flasque resolution

$$1 \longrightarrow S \longrightarrow H \longrightarrow G \longrightarrow 1 .$$

The torus S upto multiplication by a quasi-trivial torus is uniquely determined by G.

X - smooth compactification of G.

S_0 = torus with character group $\mathrm{Pic}(X(\bar{k}))$

Borovoi - Kunyavskii show that <u>S_0 is flasque</u>

Colliot-Thélène shows that $S \times$ quasi-trivial torus $\simeq S_0 \times$ quasi-trivial torus.

Let

$$1 \longrightarrow S \longrightarrow H \longrightarrow G \longrightarrow 1$$

be a flasque resolution of a connected reductive group G. Let

$$1 \longrightarrow H_1 \longrightarrow H \longrightarrow P \longrightarrow 1$$

be exact with H_1 simply connected and P quasitrivial.

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & H_1 \arrow[d] & & \\
1 \arrow[r] & S \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r] & 1 \\
 & & P \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}$$

We get maps

1) $G(k) \longrightarrow H^1(k, S)$

2) $H^1(k, G) \longrightarrow \ker\left(H^2(k, S) \longrightarrow H^2(k, P)\right)$

Since S is flasque, 1) yields a map

1') $G(k)/R \longrightarrow H^1(k, S)$

Special fields

Let k be a field with

1) $cd(k) \leq 2$

2) index = exp for csa $/k$.

Let G be a connected reductive group $/k$.
If G^{sc} has a factor of E_8, we further assume that

3) $cd(k^{ab}) \leq 1$

(CT) The maps

$$G(k)/R \longrightarrow H^1(k, S)$$

$$H^1(k, G) \longrightarrow \ker(H^2(k, S) \longrightarrow H^2(k, P))$$

induced by the flasque resolution above are ~~isomorphisms~~ bijections.

Corollary (CT - Gille - —) k 2-dimensional strict henselian field. Then for a connected linear alg. group G/k, $G(k)/R$ is finite.

Let k be a 2-dimensional strict henselian field., G connected reductive group $/k$.

Let $S \subset \Omega_k$ be a finite set of discrete valuations of k. The flasque resolution yields a map

$$A_{S,G} \longrightarrow \operatorname{coker}\left(H^1(k,S) \longrightarrow \prod_{v \in S} H^1(k_v, S)\right)$$

(CT-Gille-P)
The above map is an isomorphism.
In particular, $A_{S,G}$ is a finite abelian group.

Zero cycles of degree 1 and rational points

60's 1. Question (Serre) Let G be a connected linear algebraic group $/k$, any field. Let X be a principal homogeneous space for G/k. Suppose $X(L_i) \neq \varphi$, $[L_i : k] = m_i$, $\Sigma \mathbb{Z} m_i = \mathbb{Z}$. Is $X(k)$ non-empty?

In other words, if X has a 0-cycle of degree 1, does X have a k-rational point?

The above question is wide open in general.

60's 2. Question (Veisfeiler) Let X be a projective homogeneous variety under a connected linear algebraic group $/k$. If X has a zero cycle of degree 1, does X have a k-rational point?

3. More general questions were posed for homogeneous ~~s~~spaces under a connected linear algebraic group – CT, Totaro.

Question 1 has a positive answer for number fields (Sansuc) (HP crucial for the proof)

Question 2 also has a positive answer for number fields (Harder)

Question 3 has a negative answer:

<u>M. Florence</u> (2004) examples over number fields, stabiliser of a point finite – negative answer to 3)

—— (2004) Question 2 has a negative answer – $G = SU(A, \sigma)$, $k = \mathbb{Q}_p(t)$ based on an earlier example due to Sridharan, Suresh, —

(Note: $cd(\mathbb{Q}_p(t)) = 3$)

Hasse principle in higher dimension

Let k be a number field, $K = k(X)$ function field of a curve X/k, X absolutely irreducible. Let G be a semi-simple simply connected linear algebraic group $/K$. Set $K_v = k_v(X)$

Conjecture (Colliot-Thélène)

The map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_k} H^1(K_v, G)$$

is injective.

(P. Gille) Conjecture true for $X = \mathbb{P}^1_k$, G/k.

(Preeti, —) Conjecture true if G/k is a classical group of 'certain' types and X any curve $/k$.

Open D/k central division algebra. Is the map

$$k(X)^*/\mathrm{Nrd}(D^*_{k(X)}) \longrightarrow \prod_v \left(k_v(X)^*/\mathrm{Nrd}(D^*_{k_v(X)}) \right)$$

injective?