Recall: $F(\underline{x}) \in \mathbb{Z}[x_0, \ldots, x_s]$, degree d

$$G(\alpha) = \sum_{\underline{x} \in \mathcal{B}B \cap \mathbb{Z}^{s+1}} e(\alpha F(\underline{x}))$$

$$\#\{\underline{x} \in \mathcal{B}B \cap \mathbb{Z}^{s+1} : F(\underline{x}) = 0\}$$

$$= \int_0^1 G(\alpha)\, d\alpha$$

$\psi(B) \nearrow \infty$ slowly

$$\mathfrak{M} = \bigcup_{\substack{0 \leq a \leq q \leq \psi(B) \\ (a,q)=1}} \mathfrak{M}(q,a)$$ Major Arcs

$$\mathfrak{M}(q,a) = \left\{ \alpha \in [0,1) : \left| \alpha - \frac{a}{q} \right| \leq \psi(B) B^{-d} \right\}$$

$$\mathfrak{m} = [0,1) \setminus \mathfrak{M}$$ Minor Arcs

$$\int_0^1 G(\alpha)\, d\alpha = \underbrace{\int_{\mathfrak{M}} G(\alpha)\, d\alpha}_{\wr} + \underbrace{\int_{\mathfrak{m}} G(\alpha)\, d\alpha}_{\wr}$$

$v_\infty \left(\prod_p v_p \right) B^{s+1-d} \approx$ "Product of local densities"

small $o(B^{s+1-d})$

IX. OTHER FIELDS.

Circle method is quite general approach. Details (usually minor arc treatment) can be sensitive to characteristics of setting, but relatively robust.

Example 1. (Number fields)

Let K be an algebraic extension of $\mathbb{Q}$, say $[K:\mathbb{Q}] = n$.

Let $K^{(l)}$ $(1 \leq l \leq r_1)$ be the real conjugates, $K^{(m)}, K^{(m+r_2)}$ $(r_1+1 \leq m \leq r_1+r_2)$ the complex conjugates $(n = r_1 + 2r_2)$.

Also, when $\gamma \in K$ let $\gamma^{(i)}$ $(1 \leq i \leq n)$ denote the conjugates of γ with $\gamma^{(i)} \in K^{(i)}$ $(1 \leq i \leq n)$.

$$\mathcal{O}_K := \text{ring of integers of } K$$

When $\gamma_j \in K$, $\alpha_j \in \mathbb{R}$ $(1 \leq j \leq n)$,

put

$$\gamma_{\underline{\alpha}} = \sum_1^n \gamma_j \alpha_j$$

$$\gamma_{\underline{\alpha}}^{(i)} := \sum_1^n \gamma_j^{(i)} \alpha_j \qquad (1 \leq i \leq n)$$

Let $\omega_1, \dots, \omega_n$ denote an integral basis of $\mathcal{O}_K$,

let δ be th different (ground ideal)

$\rho_1, \dots, \rho_n$ basis of δ^{-1}

- satisfies $\displaystyle\sum_{l=1}^n \rho_i^{(l)} \omega_j^{(l)} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

<u>Detector function</u>:

$$E(\gamma_{\underline{\alpha}}) = e\left(\sum_{l=1}^n \gamma_{\underline{\alpha}}^{(l)} \right)$$

$e(z) = e^{2\pi i z}$

$$\Downarrow$$

$$\int_{[0,1)^n} E(\lambda \rho_{\underline{\alpha}})\, d\underline{\alpha} = \begin{cases} 1, & \lambda = 0 \\ 0, & \lambda \in \mathcal{O}_K^{\times}. \end{cases}$$

Now consider a bounded region $\mathcal{B} \subseteq K^{s+1}$ (e.g. relative to coordinate basis), and

$$F(\underline{x}) \in \mathcal{O}_K[x_0, \dots, x_s].$$

Define

$$G(\underline{\alpha}) = \sum_{\underline{\lambda} \in B\mathcal{B} \cap \mathcal{O}_K^{s+1}} E(F(\underline{\lambda}) \rho_{\underline{\alpha}}).$$

Then

$$\#\{\underline{\lambda} \in B\mathcal{B} \cap \mathcal{O}_K^{s+1} : F(\underline{\lambda}) = 0\}$$

$$= \int_{[0,1)^n} G(\underline{\alpha})\, d\underline{\alpha}$$

Now must define major arcs.

Given $\gamma \in K$, can associate integral ideals $\mathfrak{a}, \mathfrak{b}$ (uniquely) via

$$\gamma\delta = \mathfrak{b}/\mathfrak{a}, \quad (\mathfrak{a}, \mathfrak{b}) = 1.$$

Now define major arcs $\mathfrak{M}$ to be union of small neighbourhoods of points $\underline{\alpha} \in [0,1)^n$ for which $\rho_{\underline{\alpha}}$ is associated to $\mathfrak{b}/\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{b}) = 1$ and $N(\mathfrak{a})$ <u>suitably</u> small

---

By (essentially) formal calculation,

$$\int_{\mathfrak{M}} G(\underline{\alpha})\, d\underline{\alpha} \sim \text{"expected product of local densities."}$$

Seek conclusions where number of variables needed does not depend on $[K:\mathbb{Q}]$

Example 2. (Function fields)

Let $\mathbb{F}_q[t]$ be polynomial ring over $\mathbb{F}_q$,
finite field of characteristic p

$\mathbb{F}_q(t)$ - field of fractions

$$\mathbb{F}_q(t)_\infty := \mathbb{F}_q((1/t)) = \{\alpha = \sum_{i=-\infty}^{n} a_i t^i : a_i \in \mathbb{F}_q\}.$$

When $\alpha \in \mathbb{F}_q(t)_\infty$, define

$$\operatorname{ord} \alpha = n \quad \text{where } n = \max_i \{a_i \neq 0\}$$

$(\operatorname{ord} 0 = -\infty)$

$$\mathbb{T} := \{\alpha \in \mathbb{F}_q(t)_\infty : \operatorname{ord} \alpha < 0\}$$

(compact additive subgroup of $\mathbb{F}_q(t)_\infty$)

Note: For each $\alpha \in \mathbb{F}_q(t)_\infty$, can write $\alpha = [\alpha] + \|\alpha\|$,
with $[\alpha] \in \mathbb{F}_q[t]$ & $\|\alpha\| \in \mathbb{T}$.

Given Haar measure $d\alpha$ on $\mathbb{F}_q(t)_\infty$,

can normalize with $\int_{\mathbb{T}} d\alpha = 1$.

There is a non-trivial character

$$e_q : \mathbb{F}_q \longrightarrow \mathbb{C}$$

defined by

$$e_q(a) = e\left(\frac{\operatorname{tr} a}{p}\right) \qquad (a \in \mathbb{F}_q)$$

$$\downarrow$$

$$e : \mathbb{F}_q(t)_\infty \longrightarrow \mathbb{C}$$

$$\alpha = \sum_{i=-\infty}^{n} a_i t^i \longmapsto e_q(\operatorname{res}\alpha) = e_q(a_{-1})$$

Fact:

$$\int_{\mathbb{T}} e(\alpha f)\, d\alpha = \begin{cases} 1, & f = 0 \\ 0, & \begin{cases} f \in \mathbb{F}_q[t] \\ f \neq 0 \end{cases} \end{cases}$$

Now consider $F(\underline{x}) \in \mathbb{F}_q[t][\underline{x}_0, \dots, x_s]$.

and define

$$G(\underline{\alpha}) = \sum_{\substack{\underline{x} \in \mathbb{F}_q[t]^{s+1} \\ \deg x_i \leq B}} e(\alpha F(\underline{x})).$$

Then

$$\#\{\underline{x} \in \mathbb{F}_q[t]^{s+1}, \ \deg x_i \leq B \ (0 \leq i \leq s) ; \ F(\underline{x}) = 0\}$$

$$= \int_{\mathbb{T}} G(\alpha)\, d\alpha$$

Need to define major arcs:

Given $g, a \in \mathbb{F}_q[t]$, with g monic, $(a,g)=1$ and $\operatorname{ord} a < \operatorname{ord} g$, define

$$\mathfrak{M}(g,a) = \{ \alpha \in \mathbb{T} : \operatorname{ord}(g\alpha - a) < \psi(B) - dB$$

where $d = \deg F(\underline{x})$ and $\psi(B) \nearrow \infty$ slowly.

Major arcs $$\mathfrak{M} := \bigcup_{\substack{a, g \in \mathbb{F}_q[t] \\ g \text{ monic},\ (a,g)=1 \\ \operatorname{ord} a < \operatorname{ord} g \\ \operatorname{ord} g \leqslant \psi(B)}} \mathfrak{M}(g,a)$$

Almost formal:

$$\int_{\mathfrak{M}} G(\alpha)\, d\alpha \sim \text{"expected product of local densities"}$$

$$\approx$$

$$(q^B)^{s+1-d}$$

Hard results here are ones independent of characteristic

Not much in literature, but fairly clear analogues of classical ($\mathbb{Q}$) results whenever $\mathrm{char}(\mathbb{F}_q) > d$ (degree of polynomials involved).

<u>Theorem:</u> (Yu-Ru Liu & W., 2006)

Independent of characteristic of $\mathbb{F}_q[t]$, one has a lower bound of expected size for number of solutions of

$$a_0 x_0^d + \cdots + a_s x_s^d = 0 \qquad (a_i \in \mathbb{F}_q[t] \text{ fixed})$$

with $\underline{x} \in \mathbb{F}_q[t]^{s+1}$, $\deg x_i \leq B$, whenever

$$s \geqslant \tfrac{4}{3} d \log d + O(d \log\log d).$$

X. MINOR ARC ESTIMATES – WEYL.

Weyl, 1916, Hardy & Littlewood, 1920's ...

Illustrate first with simplest example

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k).$$

$e(z) = e^{2\pi i z}$

Since (right now) we now nothing except when α is close to a rational with small denominator, let's try to simplify.

$$|f(\alpha)|^2 = f(\alpha) f(-\alpha)$$

$$= \sum_{1 \leq x \leq P} \sum_{1 \leq y \leq P} e(\alpha(y^k - x^k))$$

$$= \sum_{1 \leq x \leq P} \sum_{1-x \leq h \leq P-x} e(\alpha((x+h)^k - x^k))$$

$y = x+h$

$$|f(\alpha)|^2 = \sum_{|h| \leq P} \sum_{x \in I(h)} e(\Delta(x^k; h)\alpha)$$

where $\quad I(h) = [1, P] \cap [1-h, P-h]$

and $\quad \Delta(x^k; h) = (x+h)^k - x^k$

$$= h\left(kx^{k-1} + \binom{k}{2} h x^{k-2} + \dots\right)$$

$$= h\, p_{k-1}(x; h)$$

So we've "simplified" exponential sum

over degree k polynomial

$$\downarrow$$

degree $k-1$ polynomial (in x)

Now iterate :

$$|f(\alpha)|^4 = \left| \sum_{|h| \leq P} \sum_{x \in I(h)} e(\Delta(x^k; h)\alpha) \right|^2$$

$$\leq (2P+1) \sum_{|h| \leq P} \left| \sum_{x \in I(h)} e(\Delta(x^k; h)\alpha) \right|^2$$

$$|f(\alpha)|^4 \leqslant (2P+1) \sum_{|h_1|<P} \sum_{|h_2|<P} \sum_{x \in I(h_1,h_2)} e(\Delta_2(x^k;h_1,h_2)\alpha)$$

where $I(h_1,h_2) = I(h_1) \cap \{x \in [1,P] : x+h_2 \in I(h_1)\}$

$$\begin{aligned}
\Delta_2(x^k;h_1,h_2) &= \Delta((x+h_2)^k;h_1) - \Delta(x^k;h_1) \\
&= (x+h_1+h_2)^k - (x+h_1)^k \\
&\quad -(x+h_2)^k + x^k \\
&= k(k-1)\,h_1h_2\,(x^{k-2} + \dots)
\end{aligned}$$

$$|f(\alpha)|^{2^j} \leqslant (2P+1)^{2^j-j-1} \sum_{\substack{|h_1|<P \\ \vdots \\ |h_j|<P}} \sum_{x \in I_j(\underline{h})} e(\Delta_j(x^k;\underline{h})\alpha)$$

with

$$\Delta_j(x^k;\underline{h}) = k(k-1)\dots(k-j+1)h_1\dots h_j(x^{k-j} + \dots)$$

Apply with $j = k-1$:

$$|f(\alpha)|^{2^{k-1}} \ll P^{2^{k-1}-k} \sum_{|h_1|<P} \cdots \sum_{|h_{k-1}|<P} \Upsilon(\underline{h})$$

where

$$\Upsilon(\underline{h}) = \sum_{x \in I_{k-1}(\underline{h})} e(\alpha \Delta_{k-1}(x^k; \underline{h}))$$

$$\leq \Big| \sum_{x \in I_{k-1}(\underline{h})} e(k!\, h_1 \dots h_{k-1} x \alpha) \Big|$$

Geometric progression in x

$$\sum_{X < x \leq X+Y} e(\alpha x) = \frac{e(\alpha[X+Y+1]) - e(\alpha(X+1]))}{e(\alpha)-1}$$

$$\ll |\sin(\pi\alpha)|^{-1}$$

$$\ll \|\alpha\|^{-1} \qquad (\alpha \neq 0)$$

$$\|\alpha\| = \min_{y \in \mathbb{Z}} |\alpha - y|$$

(trivial estimate is Y)

So

$$|f(\alpha)|^{2^{k-1}} \ll P^{2^{k-1}-k} \sum_{\substack{\underline{h} \\ |h_i|<P}} \min\{P, \|k! h_1 \dots h_{k-1} \alpha\|^{-1}\}$$

$$\ll P^{2^{k-1}-k} \Big(P^{k-1} + P^{\varepsilon} \sum_{1 \le n \le k! P^{k-1}} \min\{P, \|n\alpha\|^{-1}\} \Big)$$

suppose $|\alpha - a/q| \le q^{-2}$ & $(a,q)=1$

$$\ll P^{2^{k-1}+\varepsilon} \left(q^{-1} + P^{-1} + q P^{-k} \right)$$

$$\Downarrow$$

$$|f(\alpha)| \ll P^{1+\varepsilon} \left(q^{-1} + P^{-1} + q P^{-k} \right)^{2^{1-k}}$$

(Weyl's inequality).

Now suppose $\alpha \in \mathfrak{m}$

i.e. whenever $|\alpha - a/q| \leq \psi(P) P^{-k}$

and $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q) = 1$

then $q > \psi(P)$.

Use Dirichlet's theorem

$\exists$ $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$ and

$$|\alpha - a/q| \leq q^{-1} \left(\psi(P) P^{-k} \right)$$

with $0 \leq a \leq q \leq \psi(P)^{-1} P^k$

$$\Downarrow$$

if $\alpha \in \mathfrak{m}$, have $q > \psi(P)$

$$\Downarrow$$

$$f(\alpha) \ll P^{1+\varepsilon} \left(q^{-1} + P^{-1} + q P^{-k} \right)^{2^{1-k}}$$

$$\ll P^{1+\varepsilon} \left(\psi(P)^{-1} + P^{-1} + (\psi(P)^{-1} P^k) P^{-k} \right)^{2^{1-k}}$$

$$\ll P^{1+\varepsilon} \psi(P)^{-2^{1-k}}.$$

Potentially as strong as

$$f(\alpha) \ll P^{1-2^{1-k}+\varepsilon}$$

but in any case need $\psi(P) > P^{\delta}$ $(\delta > 0)$ to say something non-trivial.

<u>Example:</u> Consider

$$F(\underline{x}) = b_0 x_0^d + \dots + b_s x_s^d$$

with $s \geqslant d2^{d-1}$, and investigate solutions of $F(\underline{x}) = 0$ with $|x_i| \leqslant B$

$$g(\alpha) = \sum_{|x| \leq B} e(\alpha x^d), \quad g_i(\alpha) = g(b_i \alpha).$$

Minor arcs (assume $\psi(B) = B$):

$$\int_{\mathfrak{m}} |g_0(\alpha) \dots g_s(\alpha)| \, d\alpha \overset{\text{Hölder}}{\leqslant} \prod_{i=0}^{s} \left(\int_{\mathfrak{m}} |g_i(\alpha)|^{s+1} d\alpha \right)^{\frac{1}{s+1}}$$

$$\leq \max_{0 \leq i \leq s} \left(\sup_{\alpha \in \mathfrak{m}} |g_i(\alpha)| \right)^{s+1}$$

$s \geqslant d2^{d-1}$!

$$\ll \left(B^{1-2^{1-d}+\varepsilon} \right)^{s+1} = o\left(B^{s+1-d} \right)$$

Given non-degenerate binary form $\Phi(x,y) \in \mathbb{Z}[x,y]$ of degree d, can obtain

$$\sum_{|x|,|y| \leq B} e(\alpha\,\Phi(x,y)) \ll B^{2+\varepsilon}\left(q^{-1} + B^{-1} + qB^{-d}\right)^{2^{2-d}}$$

$$|\alpha - a/q| \leq q^{-2}, \quad (a,q)=1 \qquad (\text{W}' \ 1999)$$

Somewhat weaker in considering

$$\Big| \sum_{\substack{x_0,\dots,x_s \\ |\underline{x}| \in B\mathfrak{B}}} e(\alpha\, F(x_1,\dots,x_s)) \Big|^{2^{d-1}} \qquad (\text{Birch, } 1961)$$

Need to consider small values mod 1 assumed by collection of $s+1$ linear forms with varying coefficients (analogous to $\sum_{1 \leq n \leq B^d} \min\{B, \|\alpha n\|^{-1}\}$).

Geometry of numbers — singular locus plays rol

XI. MINOR ARC ESTIMATES

– MEAN VALUES.

If diophantine problem has separable structure, can apply Hölder + mean values:

$$h(\alpha) = \sum_{x \in \mathcal{A}} e(\alpha x^k)$$

$$b_0 x_0^k + \dots + b_s x_s^k = 0$$

$$\Updownarrow$$

$$\int_{\mathfrak{m}} |h(\alpha)|^{s+1} \, d\alpha \quad \text{important.}$$

$$\wedge$$

$$\sup_{\alpha \in \mathfrak{m}} |h(\alpha)| . \int_0^1 |h(\alpha)|^s \, d\alpha$$

But if $s = 2t$ is even,

$$\int_0^1 |h(\alpha)|^{2t} d\alpha = \int_0^1 h(\alpha)^t h(-\alpha)^t \, d\alpha$$

$$= \#\{ x_1^k + \dots + x_t^k = y_1^k + \dots + y_t^k ; \ x_i, y_i \in \mathcal{A}$$

Variant of Weyl's approach (Hua, 1938)

$\mathcal{A} = [1, P]$: $\int_0^1 |h(\alpha)|^{2^k} d\alpha \ll P^{2^k - k + \varepsilon}$

Industry with $\mathcal{A} = \{ n \in [1, P] \cap \mathbb{Z} : p \mid n \Rightarrow p \leq P^{\eta} \}$

$\eta > 0$ small.

$$\int_0^1 |h(\alpha)|^{2t} d\alpha \ll_{\varepsilon, \eta} P^{\lambda_t + \varepsilon}$$

$\lambda_t = 2t - k$, $t \geqslant k(\log k + \log\log k + O(1))$

$\lambda_t = 2t - k + ke^{1 - 2t/k}$ $(t \geqslant 2)$

$\lambda_t = t + \delta_t$, $\delta_t < 4k^{1/2} e^{-\frac{16k}{e^2 t}}$

very close to t

Note:

$$\int_0^1 |h(\alpha)|^{2t} d\alpha = \#\{ x_1^k + \dots + x_t^k = y_1^k + \dots + y_t^k ;\ x_i, y_i \in \mathcal{A} \} \gg P^t \quad (x_i = y_i)$$

" square-root " cancellation ?

Work of Browning, Heath-Brown, Salberger

$$\int_0^1 \Big| \sum_{1 \le x \le P} e(\alpha x^k) \Big|^6 \, d\alpha \sim 3! \, P^3$$

for $k \geq 20$?

These mean-value estimates can be converted to Weyl estimates via large sieve:

$$\sup_{\alpha \in \mathfrak{m}} |h(\alpha)| \ll P^{1 - \frac{1}{(1+o(1)) k \log k}}$$

<u>Conjecture:</u>

$$\int_0^1 \Big| \sum_{1 \le x \le P} e(\alpha x^k) \Big|^{2t} \, d\alpha \ll P^{t+\varepsilon}$$

$1 \leq t \leq k$

$$\Downarrow$$

asymptotic formula (HP & WA) for

$$b_0 x_0^k + \dots + b_s x_s^k = 0$$

for $s \geq 2k$.

MYTH # 2.

Circle method only works if number of variables is large ($2 \times \sum$ degrees).

Idea: Parameterise solution space and work inside ~~this~~ this space

Example: One can reformulate Gower's / Green-Tao as a pure application of circle method.

$$\sum_{i=0}^{s} a_{ij} x_j = b_i \quad (1 \leq i \leq r)$$

$$(s \geq r+1)$$

Solution set can be parametrised in $s+1-r$ (homogeneous) parameters - do Fourier analysis in this set (affine)

Limited by $\sqrt{}$-cancellation in this set.

<u>Note:</u> One can incorporate this idea with descent / torsors

(see E. Peyre)

<u>cf.</u> Heath-Brown, Skorobogatov ...

$$t^a (1-t)^b = \alpha \, N_{K/\mathbb{Q}}(\underline{x})$$

$$\rightsquigarrow$$

$$A \, N_{K/\mathbb{Q}}(\underline{y}) + B \, N_{K/\mathbb{Q}}(\underline{z}) = C z^d$$

$$\# \left\{ N_{K/\mathbb{Q}}(\underline{x}) = n \neq 0 : |\underline{x}| \leq B \right\} \ll (nB)^{\varepsilon}.$$

$$\Downarrow$$

$$\int_0^1 \Big| \sum_{|\underline{x}| \leq B} e(\alpha N_{K/\mathbb{Q}}(\underline{x})) \Big|^2 d\alpha \ll B^{d+\varepsilon}$$

$[K:\mathbb{Q}] = d.$