Why are Word Problems so Darned Hard?

Alan H. Schoenfeld
University of California
Berkeley, CA, USA
Alans@Berkeley.edu

Preface: Why middle school math is important…

My friend complains to a hospital doctor that his pain medicine isn't adequate.

"What's your dose?"

"Two pills every two hours."

"OK, we'll double it to four pills every four hours."

A simple "model" of solving word problems

Read problem ➲

Make a model or diagram ➲

Write equation(s) ➲

Solve equations➲

Check.

It’s not that simple.

I’m going to unpack each of the steps in the previous slide, and add a few hidden steps, to suggest why things are as hard as they are.

Reading & Language Comprehension Issues, 1.

"A five-pound box of sugar costs $1.80 and contains 12 cups of sugar. Marella and Mark are making a batch of cookies. The recipe calls for 2 cups of sugar. Determine how much the sugar for the cookies costs."

- Note: "batch" = "recipe"!

Reading & Comprehension, 2

The upper Angel Falls, the highest waterfall on Earth, are 750 m higher than Niagara Falls. If each of the falls were 7 m lower, the upper Angel Falls would be 16 times as high as Niagara Falls. How high is each waterfall?

- Hmmm… Upper falls, lower falls, Niagara falls, Waterfalls…

R&C, 3 (From a mainstream text)

The Java Joint wishes to mix organic Kenyan coffee beans that sell for $7.25 per pound with organic Venezuelan beans that sell for $8.50 per pound in order to form a 50 pound batch of Morning Blend that sells for $8.00 per pound. How many pounds of each type of bean should be used to make the blend?

Also: “savings bonds,” “fungicide,” “red pigment,” “processing a 24-exposure roll of film,” …

Mathematical Disposition (predilection to think mathematically)

A fairly typical student reads this task:

A dragonfly, the fastest insect, can fly a distance of 50 feet in about 2 seconds. How long will it take for the dragonfly to fly 375 feet?

What will the student do/say?

Here's what happened…

Less than 1/5 second pause after reading problem…

S: So, first I'll divide 375 with 50, and then – wait. Or, I will multiply… 50, no wait, now what? This is dividing…5 times what can get 8?

T: So you're thinking divide…

S: I'm not understanding. Do you look at 5 times the number first or is it the big number this is 50 into it first?

T: Well let's see, what are the quantities we're looking at here?… And what are you trying to find out?

S: (Couldn't say what the goal was:) Trying to find out how many seconds the dragonfly can fly in 375 feet… wait… How many seconds will it take it to fly 375 feet?

T: OK, … why don't you draw a picture of what you think is going on, it might be helpful…

S: (Draws a picture of a road, a town, a little dragonfly.)

T: (Focuses on quantities and where they are in the picture…)

T: Refocuses S on what they want to find out, how long to get to 375 feet) and what they know (50 feet in 2 seconds).

T: So that looks great. So what do you think you should do next?

S: I have an idea, maybe 50 times 375 divided by 2?… That won’t work. (Does computation and sees it goes nowhere.)…

T: …What are we trying to find out?

S: How many seconds will it take the dragonfly to fly 375 feet?

T: And we know what?

S: It can fly 50 feet in 2 seconds.

T: All right, what do you think, well, if it could go fly 50 feet…/

S: /in 4 seconds it would be 100 feet, in 6 it would be 200, 8 would be 300, so 9 would be 350, there's 25 missing, so 1/2 of it to get 375 so 9 1/2 seconds to get 375.

T: That sounds pretty good…

[And further exchange, where the teacher works to get the student, who now "gets" the problem, to see that rate, or proportional reasoning, can be useful.]

For many kids, math is not about sense-making.

"John wants to make wooden bookcases that are two feet wide. He has two five-foot long boards. How many two-foot long boards can he cut from them?"

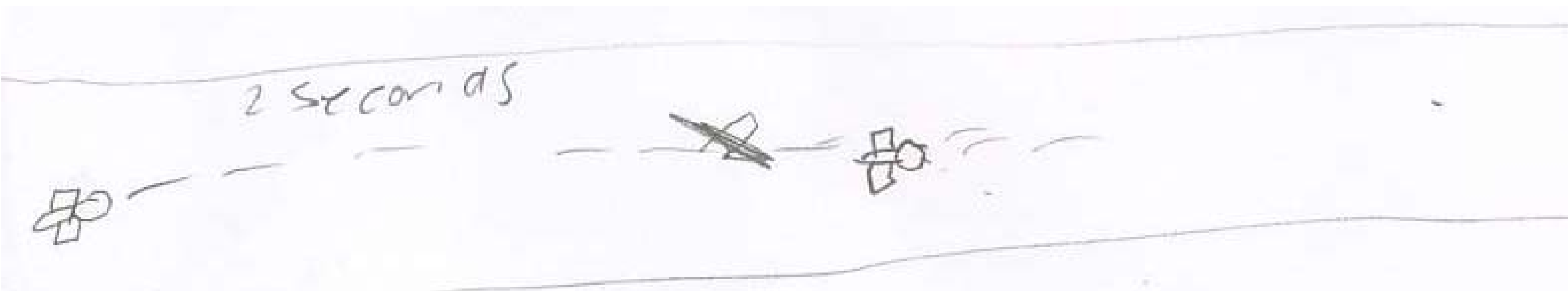
70% of the kids say "five."

Make a model or diagram

What’s relevant?

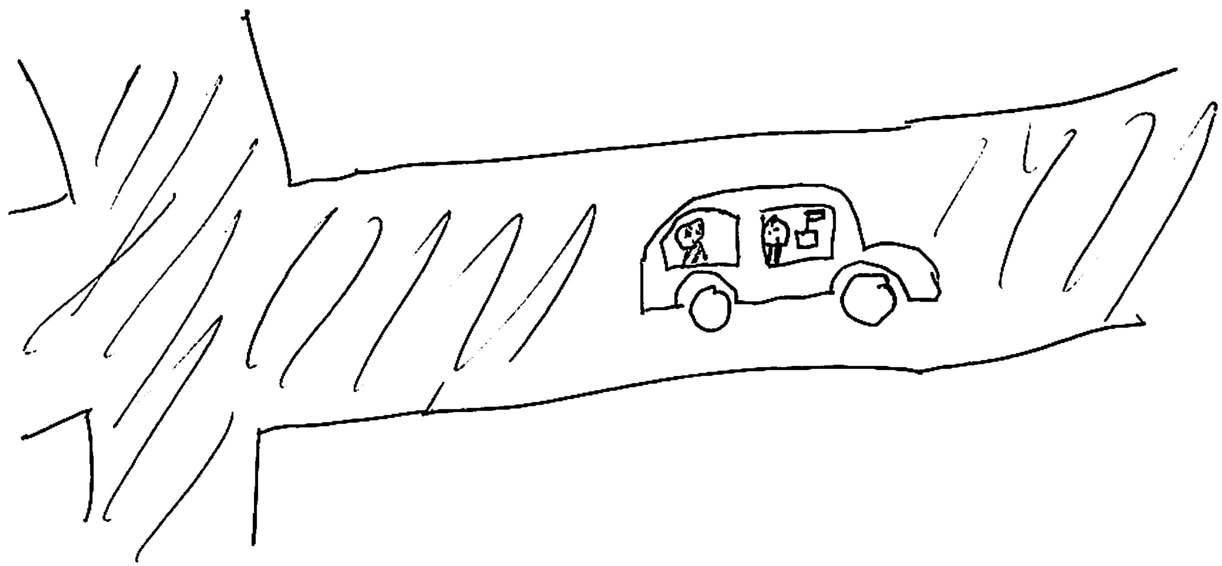
How do you picture it?

The Dragonfly Problem

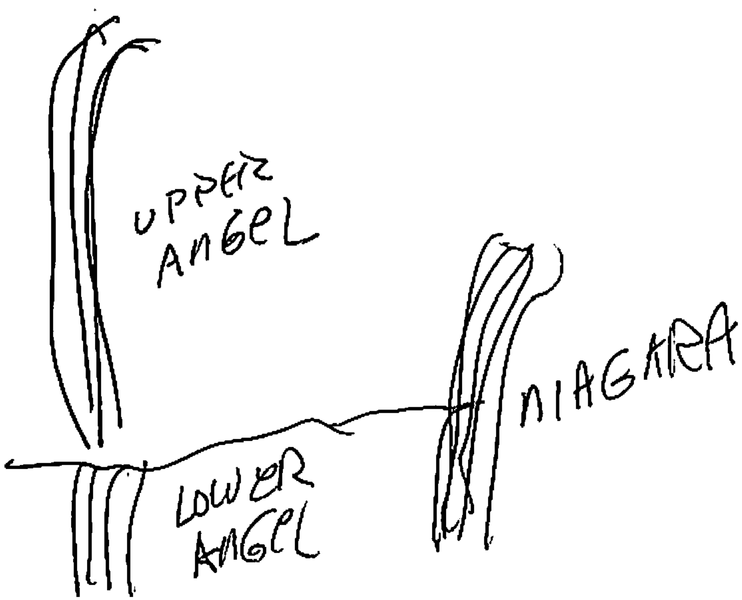


The local cab company charges $1.25 for the first mile traveled and then $0.35 for each additional mile.

Natalie spent $7.20 on a ride in a cab. How many miles did Natalie travel?



The waterfalls problem



“Situation Models” of the problem contexts

The Taxicab problem:

“The local cab company charges $1.25 for the first mile traveled and then $0.35 for each additional mile. Natalie spent $7.20 on a ride in a cab. How many miles did Natalie travel?”

The State Fair problem:

“Jamal went to the State Fair. It cost $5 to get into the fair. Each ride cost $2. If Jamal spent a total of $21, how many rides did he go on?”

Reading the math from the problem statement.

The upper Angel Falls, the highest waterfall on Earth, are 750 m higher than Niagara Falls. If each of the falls were 7 m lower, the upper Angel Falls would be 16 times as high as Niagara Falls. How high is each waterfall?

What are your first thoughts?

Mine were:

It’s two simultaneous equations in 2 variables. I’ll use convenient letters: A for Angel, N for Niagara, and derive the equations from statements.

"A river steamer…"

After hearing the three words "a river steamer" from a river current problem, one subject said, "It's going to be one of those river things with upstream, downstream, and still water. You are going to compare times upstream and downstream – or if the time is constant, it will be distance." ...

After hearing five words of a triangle problem, one subject said, "this may be something about 'how far is he from his goal' using the Pythagorean theorem." (Hinsley, Hayes, & Simon, 1977, p. 97).

Think about how that helps you with:
“The upper Angel Falls, the highest waterfall on Earth, are 750 m higher than Niagara Falls. If each of the falls were 7 m lower, the upper Angel Falls would be 16 times as high as Niagara Falls. How high is each waterfall?”

- Use A for the height of Angel Falls
- Use N for the height of Niagara Falls
  - $A = N + 750$
  - $A - 7 = 16 (N - 7)$
- Then solve.
  - Knowing where I’m going cuts through tons of stuff!

People's Conceptual Models of the situations to be analyzed

T: …What are we trying to find out?

S: How many seconds will it take the dragonfly to fly 375 feet?

T: And we know what?

S: It can fly 50 feet in 2 seconds.

T: All right, what do you think, well, if it could go fly 50 feet…/

S: /in 4 seconds it would be 100 feet, in 6 it would be 200, 8 would be 300, so 9 would be 350, there's 25 missing, so 1/2 of it to get 375 so 9 1/2 seconds to get 375.

T: That sounds pretty good…

[And further exchange, where the teacher works to get the student, who now "gets" the problem, to see that rate, or proportional reasoning, can be useful.]

A clash in perspectives

- We see this as a multiplicative/proportional situation, in which *distance per second* is a convenient and powerful unit.
- Many students see this as an additive situation, with a non-standard yardstick: every 2 seconds the dragonfly advances 50 feet.
- The students' perspective makes sense!
- But how do you lead them to the more powerful mathematical perspective?

And there's meta-level knowledge:

- Consider this problem:
  - Alan can mow the lawn in 40 minutes.
  - David can move the lawn in 50 minutes.
  - How long does it take both of them to mow the lawn together (assuming two mowers, no crashes, etc.)
- How do you approach this?
  - You have to know that the only things you can combine are rates (how much of the lawn each gets mowed per unit time.)

Deriving The Right Equation(s)

There's an art to picking the right variable, and it's not easy to pick up. Consider:

"The length of a rectangle is five inches longer than twice the width of the rectangle. The perimeter of the rectangle is 112 inches. What are the dimensions (the length and the width) of the rectangle?"

What's your choice of independent variable? How did you know to pick that one?

Solving the Equation(s)

Whew! We finally get something straightforward and procedural!

(Well, almost. Strategy choice is actually non-trivial. When to substitute, when to eliminate?…)

Checking the answers and the context.

Can there really be 31 remainder 12 buses? 1-foot-long shelves in a 2-foot bookcase?

Finishing up properly is every bit as much about sense-making as getting started.

In conclusion:

What may have seemed straightforward:

Read problem ➲

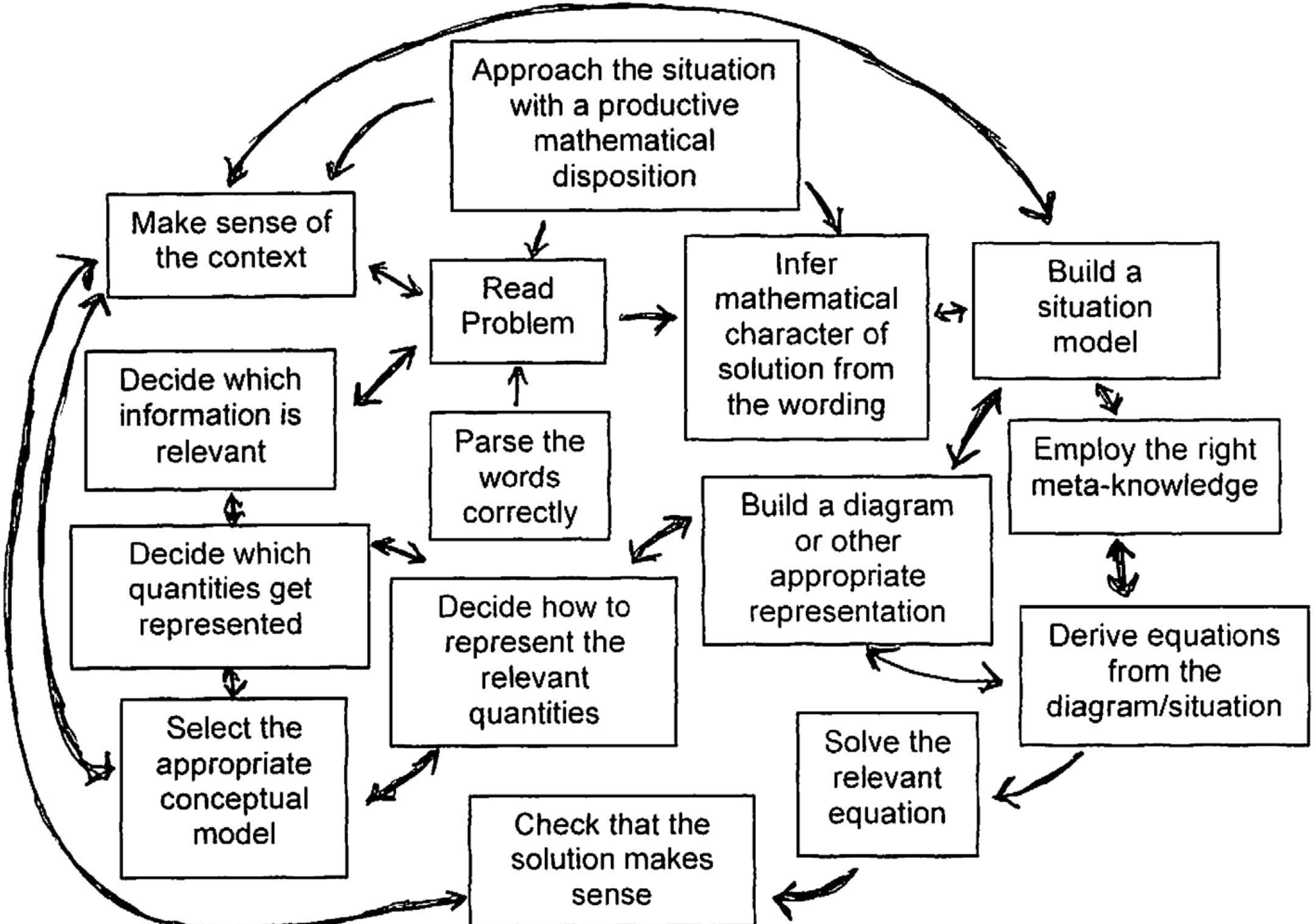
Make a model or diagram ➲

Write equation(s) ➲

Solve ➲

Check,

Actually turns out to be pretty complex:


Approach the situation with a productive mathematical disposition
Make sense of the context
Read Problem
Infer mathematical character of solution from the wording
Build a situation model
Decide which information is relevant
Parse the words correctly
Employ the right meta-knowledge
Build a diagram or other appropriate representation
Decide which quantities get represented
Decide how to represent the relevant quantities
Derive equations from the diagram/situation
Select the appropriate conceptual model
Solve the relevant equation
Check that the solution makes sense

And, every one of the boxes in that diagram requires complex pedagogical strategies and interventions, if you want there to be a good chance that students will “get it”!

And *that's* why it's hard. But, knowing that also gives us an agenda to work on.

So…

Let’s keep our sleeves rolled up and continue working…