APPLICATIONS OF GROUPS AND ISOMORPHIC GROUPS TO TOPICS IN THE STANDARD CURRICULUM, GRADES 9–11: PART II

Many relationships between groups and topics of secondary school mathematics are shown by the author, who proposes that the study of groups be included as standard fare in the mathematics curriculum of the average college-bound student.

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IT IS possible to form a set in an infinite additive group by beginning with any nonzero real a and adding it to itself over and over again, then including zero and the opposites of all numbers in the set. We call such a set the set of integral multiples of a. If $a = 3$, then here is such a set.

$$\{0, 3, -3, 6, -6, 9, -9, \cdots\}$$

With addition, this set forms a group, the group of *integral multiples* of 3.

In Part I of this article, the set of integral powers of 2 was seen to form a group with multiplication. This set can be thought of as beginning with 2, multiplying 2 by itself over and over again, then including 1 and the reciprocals of the numbers in the set. Thus the sets of integral powers and integral multiples are formed by analogous means, with the only differences being the number begun with (the *generator*) and the operation used. This hints at the existence of isomorphic groups. Listing a possible correspondence between the sets shows that this is the case.

⟨set of integral multiples of 3, +⟩	⟨set of integral powers of 2, ·⟩
0	1
3	2
−3	.5
6	4
−6	.25
9	8
−9	.125
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Part I of this article, which appeared in the February issue of the MATHEMATICS TEACHER, contains applications of groups to sentence solving, systems, and real- and complex-number operations. Definitions and examples of groups and isomorphic groups are given there.

The reader is requested to add any two numbers in the left column, multiply the corresponding numbers in the right column, and check that the answers correspond, thus verifying the isomorphism. The reader should also *subtract* one number from another in the left column, *divide* the corresponding numbers in the right column, and again check that the answers correspond.

This particular example of isomorphic groups involving multiples and powers is very good for giving the idea of what is meant by an isomorphism, but the specific numbers can disguise the applications. It is easier to work in general terms.

Application 8: Isomorphic groups of multiples and powers can be used to help students understand fundamental properties of powers.

We begin by examining the additive group of multiples. Let us suppose that the group is generated by the number a. Then an element of the set of multiples will be of the form ma, where m is an integer. Then:

1. Closure of multiples is indicated by the distributive property, $ma + na = (m + n)a$.
2. The identity multiple occurs when m is 0, $0a = 0$.
3. Inverse multiples are of the form ma and $(-m)a$.

Most students are familiar with these three properties (even if not the present context) before they have worked much with powers. But now let us consider the corresponding properties for the multiplicative group of powers. If the group is generated by x, where $x \neq 1$ and $x \neq 0$, then an element of the set of powers will be of the form x^m, where m is an integer. Since the two groups are isomorphic, for each property of multiples there will be a corresponding property of powers.

1. Closure of powers is indicated by the power property: $x^m \cdot x^n = x^{m+n}$.
2. The identity power occurs when m is 0: $x^0 = 1$.
3. Inverse powers are of the form x^m and x^{-m}.

For most students, properties of powers seem unrelated to properties of multiples. But, in fact, *every* property of multiples has a corresponding property of powers.

4. The multiple of a multiple is a multiple:

$$n(ma) = (nm)a$$

5. The multiple of a sum is the sum of the multiples:

$$m(a + b) = ma + mb$$

4. The power of a power is a power:

$$(x^m)^n = x^{mn}$$

5. The power of a product is the product of the powers:

$$(xy)^m = x^m y^m$$

This approach to properties of powers can help the student realize that properties of powers are naturally connected with multiplication and that any connections of powers and addition are tenuous. (Thus $(x + y)^2 = x^2 + y^2$ is unreasonable.) If a student thinks that zero or negative powers are unnatural, he should be reminded that they are no more unnatural than zero or negative multiples. Advanced students can be shown that property 2 can be proved from property 1 using corresponding proofs.