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Conceptions of School Algebra and Uses of Variables

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WHAT IS SCHOOL ALGEBRA?

ALGEBRA is not easily defined. The algebra taught in school has quite a different cast from the algebra taught to mathematics majors. Two mathematicians whose writings have greatly influenced algebra instruction at the college level, Saunders Mac Lane and Garrett Birkhoff (1967), begin their *Algebra* with an attempt to bridge school and university algebras:

> Algebra starts as the art of manipulating sums, products, and powers of numbers. The rules for these manipulations hold for all numbers, so the manipulations may be carried out with letters standing for the numbers. It then appears that the same rules hold for various different sorts of numbers . . . and that the rules even apply to things . . . which are not numbers at all. An algebraic system, as we will study it, is thus a set of elements of any sort on which functions such as addition and multiplication operate, provided only that these operations satisfy certain basic rules. (P. 1)

If the first sentence in the quote above is thought of as arithmetic, then the second sentence is school algebra. For the purposes of this article, then, school algebra has to do with the understanding of "letters" (today we usually call them *variables*) and their operations, and we consider students to be studying algebra when they first encounter variables.

However, since the concept of variable itself is multifaceted, reducing algebra to the study of variables does not answer the question "What is school algebra?" Consider these equations, all of which have the same form—the product of two numbers equals a third:

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot (1/n)$
5. $y = kx$

Each of these has a different feel. We usually call (1) a formula, (2) an equation (or open sentence) to solve, (3) an identity, (4) a property, and (5) an equation of a function of direct variation (not to be solved). These different names reflect different uses to which the idea of variable is put. In (1), A, L, and W stand for the quantities area, length, and width and have the feel of knowns. In (2), we tend to think of x as unknown. In (3), x is an argument of a function. Equation (4), unlike the others, generalizes an arithmetic pattern, and n identifies an instance of the pattern. In (5), x is again an argument of a function, y the value, and k a constant (or parameter, depending on how it is used). Only with (5) is there the feel of "variability," from which the term *variable* arose. Even so, no such feel is present if we think of that equation as representing the line with slope k containing the origin.

Conceptions of variable change over time. In a text of the 1950s (Hart 1951a), the word *variable* is not mentioned until the discussion of systems (p. 168), and then it is described as "a changing number." The introduction of what we today call variables comes much earlier (p. 11), through formulas, with these cryptic statements: "In each formula, the letters represent numbers. *Use of letters to represent numbers is a principal characteristic of algebra*" (Hart's italics). In the second book in that series (Hart 1951b), there is a more formal definition of variable (p. 91): "A variable is a literal number that may have two or more values during a particular discussion."

Modern texts in the late part of that decade had a different conception, represented by this quote from May and Van Engen (1959) as part of a careful analysis of this term:

> Roughly speaking, a variable is a symbol for which one substitutes names for some objects, usually a number in algebra. A variable is always associated with a set of objects whose names can be substituted for it. These objects are called values of the variable. (P. 70)

Today the tendency is to avoid the "name-object" distinction and to think of a variable simply as a symbol for which things (most accurately, things from a particular replacement set) can be substituted.

The "symbol for an element of a replacement set" conception of variable seems so natural today that it is seldom questioned. However, it is not the only view possible for variables. In the early part of this century, the formalist school of mathematics considered variables and all other mathe-