A Great Math Circle

This presentation outlines a great math circle!!!

*Caution – using this material on friends and students leads to AH-HA moments and pure math fun!

Background

- This material has been used in Chicago and Indiana
  - High school age and higher
  - This generalized problem is very difficult
  - Zero prerequisites needed for these problems
- 10 people helped me with this material over the last few years – no one does math alone so thank you!

Goals of this talk

- Everything you need to do this circle is in this presentation! Please use it.
- Go over ”The Great Circle”
    - Pigeon-Hole Principle
    - The Problem!
    - The Problem in a generalized case
    - I have the answers to the problems in the presentation for your reference

Goals of this talk, cont...

- Engaging Students with the material
- Common reactions to the problem
- References for this Circle

STEP ONE

- Go Over ”The Great Circle”
    - Pigeon-Hole Principle
        - Warm Up Problem
        - Harder Problems

A Warm Up

In a box there are 10 black socks and 12 blue socks and you need to get one pair of socks of the same colour. Supposing you can take socks out of the box only once and only without looking, what is the minimum number of socks you'd have to pull out at the same time in order to guarantee a pair of the same color?

Warm Up – The Answer

The correct answer is three. To have at least one pair of the same colour (m = 2 holes, one per colour), using one pigeonhole per colour, you need only three socks (n = 3 objects). In this example, if the first and second sock drawn are not of the same colour, the very next sock drawn would complete at least one same-colour pair. (m = 2)

Harder Problems

- (1) A busy airport sees 1500 takeoffs per day. Prove that there are two planes that must take off within a minute of each other.
- (2) One hundred points are given inside a cube of side length one. Prove that there are four of them that span a tetrahedron whose volume is at most 1/99

Harder Problems - Answers

- (1) There are 1440 minutes per day. If our 1440 minutes are the boxes, and our 1500 planes are the balls, the pigeon-hole principle says that there are two balls in the same box, that is, there are two planes that take off within a minute of each other.

Harder Problems - Answers

- (2) Split the cube into 33 prisms by planes that are parallel to its base and are at distance 1/33 from each other. By pigeon-hole principle, one of these prisms must contain four of our points. The volume of the tetrahedron spanned by these four points is at most one third of that of the prism, and the statement follows.

The Punch Line

If k + 1 or more pigeons are distributed among k pigeonholes, then at least one pigeonhole contains two or more pigeons.

Step Two: The Problem!

- There are 6 people at a party
- Rules:
  - People who know each other do not shake hands
  - People who do not know each other shake hands
- Is it possible to have a configuration where either 3 people have or have not shaken hands with each other?

The Answer to the Problem!

- Draw a graph where each person is a node
- Draw lines connecting people who shook hands because they know each other. Use blue for these connections
- Draw lines connecting people who did not shake hands because they do not know each other. Use red for these connections

The Answer to the Problem!

- Reduce the problem to a graph of three nodes with red or blue connections
- Show that it is impossible to have a graph without either a blue or a red triangle
- This proves that either 3 people must shake hands or 3 people must not shake hands

Step Three: Generalized It

- Ramsey’s theorem. Let G be a graph. A clique in G is a subgraph in which every two nodes are connected by an edge. An anti-clique, also called an independent set, is a subgraph in which every two nodes are not connected by an edge. Show that every graph with n nodes contains either a clique or an anti-clique with at least 1/2lg n nodes.

General Answer

- Proof by construction: Make a space for two piles of nodes, A and B. Then, starting with the entire graph, repeatedly add each remaining node x to A if its degree is greater than one half the number of remaining nodes and to B otherwise, and discard all nodes to which x isn’t (is) connected if it was added to A(B). Continue until no nodes left.

General Answer cont…

At most half of the nodes are discarded at each of these steps, so at least lg n steps will occur before the process terminates. Each step adds a node to one of the piles, so one of the piles ends up with at least ½ lg n nodes. The A pile contains the nodes of a clique and the B pile contains the nodes of an anti-clique.

Last Thoughts...

- Engaging Students with the material
  - Let them think about it a lot
  - Ask them how you can model the problem
- Common reactions to the problem
  - ”This is a problem for computers”
    - Ask them to write the algorithm in pseudo-code
    - If students have coding skills they often try this
  - Often people try Brute Force – let them
    - Ask how many possibilities are there? This can lead to tangent on Combinations and Permutations

References

- Introduction to the Theory of Computation by Michael Sipser, 2nd Edition
  - The Generalized problem and solution are taken from this book
- A Walk Through Combinatorics by Miklos Bona
  - Pigeon Hole Principle problems were taken from this book ch1
- General topic is Ramsey Numbers
- Sock problem is from Wikipedia
- I can be reached at hamcferron@yahoo.com

End Quote

> “ Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens. ”

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