NOTES ON THIN GROUPS

PETER SARNAK

Prepared for the MSRI hot topics Workshop on superstrong approximation (FEB 2012). My aim is to give an over-view of the developments in the theory and its applications. Naturally there is some overlap with Lubotzky's recent Colloquium Lectures [Lu 1]. See also Green's note [Gr] on group theoretic combinatorics that is closely related to some of what is disussed below.

SECTION 1 THE FUNDAMENTAL EXPANSION THEOREM

The Chinese Remainder Theorem for $SL_n(\mathbb{Z})$ asserts among other things that for $q \geq 1$ the reduction $\pi_q : SL_n(\mathbb{Z}) \longrightarrow SL_n(\mathbb{Z}/q\mathbb{Z})$ is onto. Far less elementary is the extension of this feature to $G(\mathbb{Z})$ where G is a suitable matrix algebraic group defined over $\mathbb{Q}$. The general form of this phenomenon for arithmetic groups is known as strong approximation and it is well understood [P-R].

There is a quantification of the above that is not as well known as it should be as it turns out to be very powerful in many contexts. We call this "superstrong" approximation and it asserts that if we choose a finite symmetric ($s \in S$ iff $s^{-1} \in S$) generating set S of $SL_n(\mathbb{Z})$, then the congruence Cayley graphs (X_q, S) form an expander family as $q \to \infty$ (see [H-L-W] for the definition and properties of expanders). Here the vertices x of the $|S|$-regular connected graph (X_q, S) are the elements of $SL_n(\mathbb{Z}/q\mathbb{Z})$ and the edges run from x to sx, $s \in S$. The proof of this expansion property for $SL_2(\mathbb{Z})$ has its roots in Selberg's 3/16 lower bound for the first eigenvalue λ_1 of the Laplacian on the hyperbolic surface $\Gamma\backslash\mathbb{H}$, Γ a congruence subgroup of $SL_2(\mathbb{Z})$, ([Se]). This bound is an approximation to the Ramanujan/Selberg Conjecture for automorphic forms on $GL_2/\mathbb{Q}$. The generalizations of the expansion property to $G(\mathbb{Z})$ where G is say a semisimple matrix group defined over $\mathbb{Q}$ is also known thanks to developments towards the general Ramanujan Conjectures that have been established ([B-S], [Cl], [Sa1]). This general expansion for these $G(\mathbb{Z})$'s also goes by the name 'property τ' for congruence subgroups [Lu2].

Let Γ be a finitely generated subgroup of $GL_n(\mathbb{Z})$ (more generally later on we allow it to be in $GL_n(K)$ where K a number field) and denote its Zariski closure; $Zcl(\Gamma)$ by G. If Γ is finite index in $G(\mathbb{Z})$ then the discussion above of strong and superstrong approximation can be applied. However if Γ is of infinite index in $G(\mathbb{Z})$ then $Vol(\Gamma\backslash G(\mathbb{R})) = \infty$ and the techniques used for both of these properties don't apply. In this case we call Γ "thin".

It is remarkable that under suitable natural hypotheses, strong approximation continues to hold in this thin context! The first result in this direction is [M-V-W], and Weisfeiler extended it much further. More recent and effective treatments of this can be found in [N1] and [L-Pi]. An example of the statement of strong approximation in this context is: suppose that $Zcl(\Gamma) = SL_n$ then there is a $q_0 = q_0(\Gamma)$ such that for $(q, q_0) = 1$, $\pi_q : \Gamma \to SL_n(\mathbb{Z}/q\mathbb{Z})$ is onto.

That the expansion property might continue to hold for thin groups was first suggested by Lubotzky in the 90's. Thanks to a number of major developments by many people ([SX], [Ga], [He], [B-G], [B-G-S1], [P-Sz], [B-G-T], [V]) the general expansion property is now known. The almost final version (almost because of the restriction that q be squarefree) is due to Salehi and Varju [S-V]:

THE FUNDAMENTAL EXPANSION THEOREM

Let $\Gamma \leq SL_n(\mathbb{Q})$ be a finitely generated group with a symmetric generating set S. Then the congruence graphs $(\pi_q(\Gamma), S)$, for q squarefree and coprime to a finite set of primes (which depend on Γ), are an expander family iff G^0 the connected component of $G := Zcl(\Gamma)$ is perfect (i.e. $[G, G] = G$). Moreover the determination of the expansion constant is in principle effective – if not feasible.

I will not review the techniques leading to the proof of this theorem (they have been discussed in many places including Kowalski and Tao's blogs) other than to point out that it involves three steps, the opening, the middle game and the endgame. The endgame establishes the expansion by combining sufficiently strong (but still quite crude) upper bounds for the number of closed circuits in these graphs with largeness properties of the dimensions of the irreducible representations of the finite groups $G(\mathbb{Z}/q\mathbb{Z})$. In some cases (indeed all for which reasonable bounds for the expansion are known) the proof involves the endgame only ([S-X], [Ga]). In the general case the upper bounds for the number of closed circuits is derived combinatorially. The opening involves showing that smaller subsets of $G(\mathbb{Z}/p\mathbb{Z})$ grow substantially when multiplied by themselves at least three times ([He] and the extensions [P-Sz] and [B-G-T]). A critical ingredient here is the 'sum-product' theorem ([B-K-T]) in finite fields. The middlegame is concerned with moderately large sets and is handled by the "flattening lemma" [B-G]. The latter also has its roots in combinatorics, namely the Balog-Szemeredi theorem ([Ba-Sz], [Go]). When q is not prime the analysis and combinatorics is far more complicated and difficult due to the many subgroups of of $G(\mathbb{Z}/q\mathbb{Z})$. It is handled in [B-G-S1] for SL_2 and [V] in general.

SECTION 2 APPLICATIONS

2.1 The affine sieve and diophantine analysis

The impetus for developing the expansion property for thin groups arose in connection with diophantine problems (in particular sieve problems for values of polynomials) on orbits of such thin groups ([B-G-S1]). Both strong approximation and superstrong approximation are crucial ingredients in executing a BRUN combinatorial sieve in this setting. The theory is by now quite advanced and in particular the basic theorem of the affine sieve has been established in all cases where it is expected to hold ([S-S]).

For various special examples such as for integral Apollonian packings (which has turned out to be one of the gems of the theory [Sa2]) much more can be said thanks to special features. Firstly in this case one can develop an archimedian count for the number of points in an orbit in a large region. This is done by combining spectral methods (using techniques which when $\Gamma\backslash SO(n-1,1)(\mathbb{R})$ is geometrically finite go back to [P], [Su] and [L-P]) with ergodic theoretic methods ([K-O], [O-S], [L-O]). For the diophantine applications one needs an archimedian spectral gap for the induced congruence groups, rather than the combinatorial expansion. [B-G-S 2] establishes the transfer of this information from the combinatorial to archimedian setting in this infinite volume case.

Two very recent highlights of these developments are the 'almost all' local to global results [B-K-1] and [B-K-2]. The first concerns integral Apollonian packings and the question is which numbers are curvatures? The expected local to global conjecture ([G-L-M-W-Y], [F-S]) is proven for all but a zero density set of integers (the Conjecture asserts that there are only a finite number of exceptions). Prior to that [B-F] had shown that the number of integers that are achieved is of positive density. The second development concerns the Zaremba problem which asserts that if $A \geq 5$, the set of integers $q \geq 1$ for which there is a $1 \leq b \leq q$, $(b, q) = 1$ and for which the coefficients of the continued fraction b/q are bounded by A, consists of all of $\mathbb{N}$. In [B-K2] the theory of thin subgroups of $SL_2(\mathbb{Z})$ is extended to thin semi(sub)groups. (one has to abandon direct spectral methods and replace them by dynamical ones [La], [B-G-S2]). They show that for $A \geq 50$ the set of exceptions to the Zaremba conjecture is of zero density in $\mathbb{N}$.

2.2 Random Elements in Γ:

It is well known that for any reasonable notion of randomness, the random $f \in \mathbb{Q}[x]$ is irreducible and has Galois

group the full symmetric group of order deg f.
In [Ri] the study of such questions for the characteristic polynomial f_γ of a random element γ in $Sp(2g,\mathbb{Z})$ and more general Γ's, was initiated. The random element in $Sp(2g,\mathbb{Z})$ is generated by running a symmetric random walk with respect to a measure μ whose support generates $Sp(2g,\mathbb{Z})$. The expansion property is used via a sieving argument to show that the probability that f_γ is reducible is exponentially small. This and some generalizations are then coupled with the theory of the mapping class group $\mathcal{G}$ to show that the random element in $\mathcal{G}$ is pseudo Anosov. These irreducibility questions and much more, are extended and refined in terms of the sieves that are applied, in the monograph [Ko]. Again strong and superstrong approximation plays a central role.

In a different direction [L-M] examine some group theoretic questions for linear groups using a random walk and a sieve. An example of what they show is: Let Γ be a finitely generated subgroup of $GL_n(\mathbb{C})$ which is not virtually solvable, then the set of proper powers $\underline{P} := \bigcup_{m=2}^{\infty} \Gamma^m$, is exponentially small (in terms of hitting $\underline{P}$ in a long random walk). In particular finitely many translates of $\underline{P}$ cannot cover Γ.

2.3 Gonality and Heegard Genus:

A compact Riemann surface of genus g can be realized as a covering of the plane of degree at most $g+1$ (Riemann-Roch). The gonality $d(X)$ of X is the minimal degree of such a realization. Unlike $g(X)$, $d(X)$ is a subtle conformal invariant. In [Z] (and later [A1]) the differential geometric inequality of [Y-Y] is extended to the setting of $X = \Gamma\backslash\mathbb{H}$ a finite area quotient (orbifold) of the hyperbolic plane. If $A(X)$ is its area and $\lambda_1(X)$ its first Laplace eigenvalue then

$$d(X) \geq \frac{\lambda_1(X)\, A(X)}{8\pi}. \qquad (1)$$

This together with the known bounds towards the Ramanujan/Selberg Conjectures for congruence (arithmetic) X's (see [B-B] for the best bounds for GL_2/K, K a number field which is what is relevant here) imply that for these X's, the ratio of any two of $d(X)$, $A(X)$ and $(g(X)+1)$ is bounded universally from above and below.

There is a generalization of (1) to finite volume quotients $X = \Gamma\backslash\mathbb{H}^m$ (orbifolds) of hyperbolic m-space [A-B-S-W]. This is stated in terms of [L-Y]'s notion of conformal volume. It gives an inequality between $\mathrm{Vol}(X)$, $\lambda_1(X)$ and the conformal volume of a piecewise conformal map of X into S^n. Again together with the known universal lower bounds for $\lambda_1(X)$ when X is congruence arithmetic ([B-S], [Cl]) gives a linear in the volume, lower bound for the conformal volume of a conformal map of X to S^m. This has a nice application to reflection groups. A discrete group of motions of $\mathbb{H}^m$ is called a reflection group if it is generated by reflections (a reflection of $\mathbb{H}^m$ is a nontrivial isometry which fixes an $m-1$ dimensional hyperplane). Using the inequalities mentioned above one shows ([L-M-R] for $m=2$ and [A-B-S-W] for $m>2$) that the set of maximal arithmetic reflection groups is finite for each m. Now Vinberg [Vi] and [P] have shown that for $m \geq 1000$ a reflection group can never be a lattice. Thus the totality of all maximal arithmetic reflection groups is finite.

(1) has interesting applications to diophantine equations. As observed in [A2] and [Fr], Faltings' finiteness theorem for rational points on subvarieties of abelian varieties [Fa] can be used to prove finiteness of rational points on curves, whose coordinates lie in the union of all number fields of bounded degree, as long as one can show the gonality of the curve is large enough. For example if $X_0(N)/\mathbb{Q}$ is the familiar modular curve of level N and if D is given, then for $N \geq 230D$(*), the set of points on $X_0(N)$ with coordinates in the union of all number of fields of degree at most D, is finite! Recently [E-H-K] have applied similar

(*) this follows from (1) and explicit Ramanujan bounds.

reasoning to a diophantine problem on a tower of curves. It arises from questions of reducibility and symmetry of specializations of members of a 1-parameter family of varieties. The curves that arise (as the parameter) are determined by the monodromy group Π of the family (see below), and it lies in $Sp(2g; \mathbb{Z})$ and is assumed to be Zariski dense in $Sp(2g)$. In order to show that the gonalities of the curves in question increase quickly enough, they use the combinatorial expansion that is provided by the Fundamental Expansion Theorem. Typically it is not known if Π is thin or not (see Section 3) but the beauty of the Fundamental Theorem is that one does not need to know!

There is an inequality similar to (1) for the Heegard genus of a hyperbolic 3-manifold X. It is known that such an X can be decomposed into two handle bodies with common boundary a surface of genus h (called a Heegard splitting). The minimal genus of such a surface in a splitting is called the Heegard genus of X which we denote by $g(X)$. Like the gonality it is a much more subtle (this time topological) invariant of X than its Volume. In [La] it shown that for complete X of finite volume

$$g(X) \geq \frac{\lambda_1(X)\,\mathrm{Vol}(X)}{32\pi} \qquad (2)$$

Applying this together with the universal lower bounds for λ_1 for congruence arithmetic X's, shows that the Heegard genus of a congruence hyperbolic three manifold is in order of magnitude a linear function of its volume. In particular any arithmetic 3-manifold has an infinite (by congruence subgroups) tower of coverings whose Heegard genus grows linearly with the volume. One can ask if the same is true for any hyperbolic 3-manifold and the answer is yes as was shown in [L-L-R]. Using local rigidity of lattices in $SL_2(\mathbb{C})$ one can realize Γ where $X = \Gamma\backslash\mathbb{H}^3$, as a finitely generated subgroup of $SL_2(K)$, where K is some number field. If Γ is not arithmetic then Γ is thin (in $SL_2(\mathcal{O}_K)$ perhaps allowing denominators at finitely many places). Since its projection on the identity embedding of $K \hookrightarrow \mathbb{C}$ is discrete. Using the fundamental expansion theorem gives a lower bound on λ_1 for a 'congruence tower' of Γ and one then applies (2).

Section 3 Ubiquity of Thin Groups

Given a finitely generated group Γ in $GL_n(\mathbb{Z})$ one can usually compute $G = Zcl(\Gamma)$ without too much difficulty. On the other hand deciding if Γ is thin can be formidable. In fact one is flirting here with questions that have no decision procedures (I thank Rivin for alerting me to these pitfalls that are close by). For example if $\Gamma = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ then there is no decision procedure to determine if an element $A \in \Gamma$ is in the group generated by a general set of say seven elements [Mi]. Even for Gromov hyperbolic groups the question of whether a finitely generated subgroup generates a finite index subgroup, has no decision procedure ([Rip], [B-M-S]). Mercifully strong and superstrong approximation only ask about $Zcl(\Gamma)$. Still one is curious about thinness when applying these theorems and sometimes for good reason. At least in the affine sieve setting, the quality of the expansion impacts the results dramatically (see [N-S] for the cases when Γ is a lattice) while the diophantine orbit problems become ones of integer points on homogeneous varieties when Γ is a lattice. Whether the typical Γ is thin or not is not so clear, and may depend on how Γ arises.

3.1 Schottky, Ping-Pong:

Schottky groups in which the generators play ping-pong ([Ti], [B-Ge]) are one of the few classes of discrete groups whose group theoretic structure is very simple. If one choses $A_1, A_2, \ldots, A_\ell$ independently and at random in $SL_n(\mathbb{Z})$ $(n \geq 2)$ then with high probability $\Gamma = \langle A_1, \ldots, A_\ell \rangle$ will be free on these generators, Zariski dense in SL_n and thin. If the A_j's are chosen at the m-th step of a μ-random walk ($m \to \infty$) and supp(μ) generating $SL_n(\mathbb{Z})$, then this was proved in [A]. A more geometric version is proven in [Fu-Ri] where the A's are chosen independently and uniformly by taking them from the set of B's with $\max(\|B\|, \|B^{-1}\|)$ less than X. Here $\| \ \|$ is any Euclidean norm on the space of matrices and $X \to \infty$. Not only is Γ thin but it is very thin in the sense that Hausdorff dimension of the limit set of Γ acting on $\mathbb{P}^{n-1}(\mathbb{R})$ is arbitrarily small.

3.2 Nonarithmetic Lattices

If $\Gamma \leq G$, with $G \not\cong SL_2(\mathbb{R})$, is an irreducible nonarithmetic lattice in a semisimple real group G, then Γ is naturally thin in the appropriate product by conjugates. The argument is the same as in the last section using local rigidity. The certificate of being thin is that Γ is discrete in the factor corresponding to G. Examples of this kind which come from monodromy of hypergeometric differential equations in several variables are given in [D-M]. It appears that these were the first examples of thin monodromy groups. Other examples in products of SL_2's are given in [N2]. These examples aren't even finitely presented. Teichmuller curves in the moduli space M_2 of curves of genus 2 give via Abel-Jacobi, curves in $\mathcal{A}_2$ whose monodromies (inclusion of fund. gps) are thin ([Mc]).

3.3 Reflection Groups in Hyperbolic Space

Let f be an integral quadratic form in n-variables and of signature $(n-1,1)$. For $n \geq 3$, $O_f(\mathbb{Z})$ the group of integral automorphs of f is a lattice in $G = O_f(\mathbb{R})$. The reflective subgroup R_f is the subgroup of $O_f(\mathbb{Z})$ which is generated by all the reflections in $O_f(\mathbb{Z})$. R_f is a normal subgroup of $O_f(\mathbb{Z})$ and if it is nontrivial then $Zcl(R_f) = O_f$. Vinberg [Vi] and Nikulin [Ni] have examined the question of when R_f is finite index in $O_f(\mathbb{Z})$ (they call such an f reflective). In particular in [Ni] it is shown that there are only finitely many f's (up to integral equivalence) which are reflective. Thus for all but finitely many f's, R_f if it is nontrivial, is a thin group in $GL_n(\mathbb{Z})$ (albeit possibly infinitely generated). Note that Nikulin's theorem fails for $n=2$. If f is a binary quadratic form then f is reflective iff it is ambiguous in the sense of Gauss (see [Sa3]) and Gauss determined the ambiguous forms in his study of genus theory.

3.4 Monodromy Groups

A very natural source and perhaps the oldest one, of finitely generated linear groups comes from monodromy in all its guises. The very classical case of Monodromy of the hypergeometric equation

${}_nF_{n-1}(\alpha,\cdot,z)$ enjoys all the interesting

features of monodromy groups of one parameter families. A detailed study of the Zariski closure of these is carried out it [B-H] and we use their notations. We are interested in the algebraic cases (and primitive ones) so the parameters $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$ are in $\mathbb{Q}^n$ with $\alpha_j,\beta_j\in[0,1)$. The hypergeometric equation is

$$D:=(\theta+\beta_1-1)(\theta+\beta_2-1)\cdots(\theta+\beta_n-1)-z(\theta+\alpha_1)\cdots(\theta+\alpha_n),\ \theta=\frac{d}{dz},$$
$$Du=0\ .$$

It is regular outside $\{0,1,\infty\}$ and at these three points it has regular singularities. The n-dimensional space V of solutions in a neighbourhood of a base point $z_0\in S=\mathbb{P}^1\setminus\{0,1,\infty\}$ may be analytically continued in S and this gives rise to a representation M of $\pi_1(S)$ in $GL_n(V)$. The monodromy group of the hypergeometric differential equation with parameters α,β is denoted $H(\alpha,\beta)$ and it is by definition the image of $\pi_1(S)$. It is the group generated by $h_0=M(\alpha,\beta)(g_0)$ and $h_\infty=M(\alpha,\beta)(g_\infty)$ where g_0 and g_∞ are loops about 0 and ∞.

We restrict our attention to H's which lie in $GL_n(\mathbb{Z})$ (after conjugation). The Zariski closure of such $H(\alpha,\beta)$'s which are automatically self dual is either finite or O_n or Sp_n [B-H]. They determine which it is and also give a complete list of all the finite $H(\alpha,\beta)$'s.

When H is not finite it is apperently quite challanging to describe $H(\alpha,\beta)$ and even to decide if it is thin or not. A very interesting case is the Dwork family; n even $\alpha = (0,0,\dots 0)$, $\beta = (\frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1})$. For $n=2$, $H(\alpha,\beta)$ is commensurable with $SL_2(\mathbb{Z})$ but for $n \geq 4$ $H(\alpha,\beta)$ is probably thin but this is not known. These monodromies were used recently with great success in [H-S-T], and what is critical to this application is that H satisfy strong approximation. The case $n=4$ arises in the study of mirror symmetry for Calabi-Yau 3-folds ([C-D-G-P] and [C-Y-Y] compute a number of other hypergeometrics) where $H(\alpha,\beta)$ is a key symmetry group of the problem. Our question here is very concrete and elementary; Do

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

generate a finite or infinite index subgroup H in $Sp(4,\mathbb{Z})$? By the congruence subgroup property H is contained in an easy to compute congruence subgroup B. Either $H=B$ or H is thin and profinitely pretends to be B, which is it?

If $Zcl(H(\alpha,\beta))$ is $O(n)$ and over $\mathbb{R}$ it has signature $(n-1,1)$, then one can bring cohomological methods as well as convexity theory in hyperbolic space, to bear on the question of thinness. Progress in this direction has been made in [F-M-S]. We call such $H(\alpha,\beta)$'s (which are also defined over $\mathbb{Q}$) for which $H(\alpha,\beta)$ is contained in $GL_n(\mathbb{Z})$), hyperbolic hypergeometrics. They only occur for n odd and there is a large finite list of

(+) based on extensive computations by Fuchs and Rivin

Sporadic examples for $n<10$. For $n>10$ they fall into 9 explicit infinite families.

For example $n \geq 3$

$$\left.\begin{aligned} \alpha &= \left(0, \tfrac{1}{n+1}, \ldots, \tfrac{n-1}{2(n+1)}, \tfrac{n+3}{2(n+1)}, \ldots, \tfrac{n}{n+1}\right) \\ \beta &= \left(\tfrac{1}{2}, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, \tfrac{n-1}{n}\right) \end{aligned}\right\} (*)$$

on of the infinite sequence of hyperbolic hypergeometrics.

What no doubt is the case is that all but finitely many of the hyperbolic hypergeometric monodromies are thin. At least this is proved for the sequence (*). One can show that the general hyperbolic $H(\alpha,\beta)$ is commensurable with a group of hyperbolic motions generated by Cartan involutions (a Cartan involution of a symmetric space is a motion which inverts geodesics in a point). For some of the families such as (*) one can apply some diophantine involutions connected with binary quadratic forms (specifically reciprocal forms [Sa3]) to relate $H(\alpha,\beta)$ to a reflection group and then use Vinberg's theory of such groups.

(*) yields in particular what appear to be the first example of thin algebraic monodromies for which $Zcl(H)$ is simple. My belief (+) is that thin is the rule in general for these $H(\alpha,\beta)$'s. That is with the exception of a finite number of (α,β)'s, all the $H(\alpha,\beta)$'s which are infinite, are thin. The trouble with the nonhyperbolic case is that there is no robust certificate for a group to be thin. Ping-pong which is apparently the only game in town, is something you can play if you can choose the players but it is hard to play if someone chooses them for you.

REFERENCES

[Gr] B. Green, "What is an approximate group" to appear in Notices of AMS.

[Lu 1] A. Lubotzky BAMS 49 (2012), 113–162

[P-R] V. Platonov and A. Rapinchuk "Algebraic groups and Number Theory" AP 1991

[H-L-W] S. Hoory, N. Linial and A. Wigderson BAMS 43 (2006) 439–561

[Se] A. Selberg 1965 Proc. Symp Pure Math Vol VIII, 1–15

[B-S] M. Burger and P. Sarnak Inv. Math 106 (1991) 1–11.

[Cl] L. Clozel Inv. Math. 151 (2003) 297–328.

[Sa 1] P. Sarnak "Notes on the generalized Ramanujan Conjectures" publications ias.edu/sarnak

[Lu 2] A. Lubotzky "What is property tau" Not. A.M.S 52 (2005) 626–627

[M-V-W] C. Matthews, L. Vaserstein and B. Weisfeiler Proc. L.M.S 48 (1984) 514–532

[N 1] M. Nori Invent. Math. 88 (1987) 257–275.

[L-Pi] M. Larsen and Pink "Finite subgroups of algebraic groups" 1995

[S-X] P. Sarnak and Xue, DMJ 64 (1991) 207–227

[Ga] A. Gamburd Israel Math. Jnl 127 (2002) 157–200

[He] H. Helfgott Ann Math. 167 (2008) 601–623

[B-G] J. Bourgain and A. Gamburd, Ann Math 167 (2008) 625–642.

[B-G-S 1] J. Bourgain, A. Gamburd and P. Sarnak, Inv. Math. 179 (2010) 559–644

[P-S] L. Pyber, E. Szabo "Growth in finite simple groups" ArXiv 1001.4556

[B-G-T] E. Breuillard, B. Green and T. Tao, "Approximate subgroups of linear groups" ArXiv 1005.1881

[V] P. Varju "Expansion in $SL_d(\mathcal{O}_K/I)$, I-square free" ArX. 1001.3664

[S-V] A. Salehi and P. Varju "Expansion in perfect groups" ArXiv 1108.4900

[Ba-Sz] A. Balog, E. Szemeredi Combinatorica 14, 263–268, 1994

[Go] T. Gowers GEOM FUNCT. ANAL. 11 465–588 (2011)

[S-S] A. Salehi and P. Sarnak "Affine linear sieve" arXiv 1109.6432

[Sa 2] P. Sarnak Am. Math. Month. 118 (2011) 291–306

[P] S. Patterson Acta Math 136 (1976) 241–273

[Su] D. Sullivan Pub IHES 50 (1979) 171–202.

[B-K-T] J. Bourgain, N. Katz, T. Tao GAFA 14 (2004) 27–57

[L-O] M. Lee and H. Oh, "Effective circle count for Apollonian Packings and closed horospheres" ArXiv 2012.

[L-P] P. Lax and R. Phillips J. Fun. Anal 46 (1982) 280-350

[K-O] A. Kontorovich and H. Oh JAMS 24 (2011) 603-648

[O-S] H. Oh and N. Shah "Equidistribution and counting for orbits of geometrically finite hyperbolic groups" preprint.

[B-G-S 2] J. Bourgain A. Gamburd P. Sarnak, ArXiv 0912.5021

[B-K 1] J. Bourgain and A. Kontorovich; in preparation.

[B-K 2] J. Bourgain and A. Kontorovich "On a conjecture of Zaremba" ArXiv 1107.3776

[G-L-M-W-Y] R. Graham, J. Lagarias, C. Mallows, R. Wilks and C. Yan, Jnl Numb. Th. 100 (2003) 1-45.

[F-S] E. Fuchs and K. Sanden Exp Math. Vol 20, 2011, 380-400

[B-F] J. Bourgain and E. Fuchs JAMS 24 (2011) 945-967.

[La] S. Lalley Acta Math 163, 1-55 (1989).

[Ri] I. Rivin D.M.J. 142 (2008) 353-379.

[Ko] E. Kowalski "The large sieve and its applications" CUP 2008

[L-M] A. Lubotzky and C. Meiri "Sieve methods in group theory powers in linear groups" ArXiv. 1107-3666

[Z] P. Zograf J. Math. Sc. 36, 1, 106-114 (1994)

[A1] D. Abramovich IMRN 20 (1996) 1005-1011

[Y-Y] P. Yang and S. Yau Ann. Scu Norm. Sup Pisa 7, 55-63 (1980)

[B-B] V. Blomer and F. Brumley Ann. Math. 174 (2011) 581-605

[A-B-S-W] I. Agol, M. Belolipetsky, P. Storm and K. Whyte, Groups Geom Dyn (2008) 481-498.

[L-Y] P. Li and S. Yau Inv. Math 69 (1982) 269-21

[L-M-R] D. Long, C. MacLachlan, A. Reid P.A.M.Q. Vol 2, 2, 1-31 (2006)

[Vi] E. Vinberg Trudy Moscow Mat. Obs. 47 (1984) 68-102

[P] M. Prokhorov Izv. Akad Nauk SSSR 50 (1986) 413-424

[A2] D. Abramovich PhD. Thesis Harvard 1991.

[Fr] G. Frey Israel Jnl of Math (85) 1994, 79-83

[Fa] G. Faltings Ann of Math 133 (1991) 549-576.

[E-H-K] J. Ellenberg, C. Hall and E. Kowalski, "Expander graphs gonality and variations of Galois representations" arXiv. 1008.3675

[La] M. Lackenby, Inv. Math. 164 (2006), 317-359

[L-L-R] D. Long, A. Lubotzky and A. Reid, J. Top. 1 (2008) 152-158

[Mi] K. Mikhailova Dok. Akad Nauk SSSR 119 (1958) 1103-1105

[Rip] E. Rips BLMS 14 (1982) 45-47.

[B-M-S] G. Baumslag, C. Miller and H. Short, BLMS 26 (1994) 97-101

[Mc] C. McMullen, JAMS Vol 16, 4, 857-885 (2003)

[N-S] A. Nevo and P. Sarnak Acta Math 205 (2010) 361-402

[Ti] J. Tits J. Alg. 20: 250-270 (1972).

[B-Ge] E. Breuillard and T. Gelander J. Alg. 26, 448-467 (2003)

[A] R. Aoun "Random subgroups of linear groups are free" ArXiv 1005.3445

[Fu-Ri] E. Fuchs and I. Rivin, in preparation.

[D-M] P. Deligne and G. Mostow, PUBL. IHES 63 (1986) 5-89

[N2] M. Nori C. R. Acad. Sci. I Math. 302 (1986) 71-73

[Ni] V. Nikulin ICM Berkeley 1986 Vol 1, 654-671.

[Sa 3] P. Sarnak "Letter to Jim Davis 2005" publications.ias.edu/sarnak

[B-H] F. Beukers and G. Heckman Inv. Math 95 (1989) 325-354

[H-S-T] M. Harris, N. Shepherd-Barron, R. Taylor, Ann Math. 171 (2010) 779-813

[C-D-G-P] P. Candelas, X de la Ossa, P. Green, L. Parkes, Nuc. Phys B 359 (1991) 21-74

[C-Y-Y] Y. Chen, Y. Yang and N. Yui J. Reine. Angew Math. 616 (2008) 167-203

[F-M-S] E. Fuchs, C. Meiri and P. Sarnak, in preparation.

Final Note: The gonality of a congruence arithmetic surface being linear in its genus and the Heegard genus of a congruence arithmetic hyperbolic 3-manifold being linear in its volume, as well as the proof that there are only finitely many maximal arithmetic reflection groups, all appeal to the uniform lower bounds for λ_1 for all such manifolds. This follows from what is known towards the Ramanujan Conjectures but it does not follow from the fundamental expansion theorem since the latter applies only to one tower at a time. As far as the general Ramanujan Conjectures some progress has been made since the report [Sa 1]. Namely in [The Endoscopic classification of representations of orthogonal and symplectic groups, J. Arthur 2011] a precise formulation of the Ramanujan Conjecture for these groups is given. Moreover it is shown (assuming the fundamental lemma forms which itself should be a theorem before too long) that these conjectures will follow, if one can prove them for GL_n.