

Cauchy Problem and Cosmic Censorship

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An *initial data set* for the Vacuum Einstein Equation consists of a connected, n -dimensional manifold M together with a Riemannian metric σ and a symmetric $(0, 2)$ tensor field K such that (M, σ, K) satisfies the Vacuum Einstein Constraint Equations:

$$\begin{cases} R_\sigma - |K|_\sigma^2 + (\operatorname{tr}_\sigma K)^2 = 0 \\ \operatorname{div}_\sigma K - d(\operatorname{tr}_\sigma K) = 0. \end{cases} \quad (1)$$

Remark: σ and K are assumed to be smooth in this lecture.

Question: Does there exist a *globally hyperbolic* spacetime (\bar{M}^{n+1}, \bar{g}) satisfying

- (1) the Vacuum Einstein Equation $\text{Ric}(\bar{g}) = 0$
- (2) there exists an isometric embedding

$$\iota : (M^n, \sigma) \longrightarrow (\bar{M}^{n+1}, \bar{g})$$

with $\iota^*(\text{III}) = K$, where III is the second fundamental form of $\iota(\Sigma)$ in (\bar{M}^{n+1}, \bar{g}) w.r.t. the future timelike unit normal

- (3) $\iota(M)$ is a Cauchy hypersurface in (\bar{M}^{n+1}, \bar{g}) ?

Recall that the domain of dependence $D(A)$ of an achronal set A in an arbitrary spacetime \bar{M} satisfies the basic property that

$\text{int}(D(A))$ is a globally hyperbolic open set in \bar{M} .

Moreover, if A is an acausal C^0 hypersurface, then $\text{int}(A)$ is an open set itself.

Suppose one already has a spacetime (\bar{M}^{n+1}, \bar{g}) satisfying conditions (1) and (2) above and suppose $i(M)$ is acausal in (\bar{M}^{n+1}, \bar{g}) . Replacing \bar{M}^{n+1} by $D(\iota(M))$, one obtains a spacetime $(D(\iota(M)), \bar{g})$ satisfying conditions (1)–(3).

When is a spacelike hypersurface an achronal/acausal set?

Lemma Let M be a spacelike hypersurface in a spacetime \bar{M} . If $\bar{M} \setminus M$ is disconnected, then M is achronal.

Lemma An achronal, spacelike hypersurface M is necessarily acausal.

Therefore, given a vacuum initial data set (M, σ, K) , if one can find a spacetime metric \bar{g} on an open set U , containing $M \times \{0\}$, in the product manifold $M \times \mathbb{R}^1$ such that

- ▶ $\text{Ric}(\bar{g}) = 0$ on U
- ▶ $\iota : M \rightarrow M \times \{0\}$ satisfies $\iota^*(\bar{g}) = \sigma$ and $\iota^*(\text{III}) = K$,

then $(D(\iota(M)), \bar{g})$ is a globally hyperbolic spacetime satisfying (1)-(3).

Analyzing the Ricci Curvature

Let \bar{g} and g be two arbitrary metrics (Riemannian or Lorentzian) on an open set W . We treat g as a background metric. Let $\bar{\nabla}$, ∇ be the covariant differentiation associated to \bar{g} , g respectively.

It is a basic fact that $A = \bar{\nabla} - \nabla$ is a tensor on W . In any local coordinate chart $\{x^\mu\}$ on W , one has

$$A_{\mu\nu}^\delta \partial_\delta := \bar{\nabla}_{\partial_\mu} \partial_\nu - \nabla_{\partial_\mu} \partial_\nu$$

where

$$A_{\mu\nu}^\delta = \frac{1}{2} \bar{g}^{\delta\gamma} (\bar{g}_{\mu\gamma;\nu} + \bar{g}_{\gamma\nu;\mu} - \bar{g}_{\mu\nu;\gamma}) \quad (2)$$

where “ ; ” denotes the covariant differentiation w.r.t the background metric g .

In terms of A , direct calculation gives

$$\text{Ric}(\bar{g})_{\mu\nu} - \text{Ric}(g)_{\mu\nu} = A_{\mu\nu;\alpha}^{\alpha} - A_{\nu\alpha;\mu}^{\alpha} + A_{\mu\nu}^{\delta} A_{\alpha\delta}^{\alpha} - A_{\mu\delta}^{\alpha} A_{\alpha\nu}^{\delta}. \quad (3)$$

In what follows, we write $h_1 \approx h_2$ to suggest that $h_1 - h_2$ is a sum of tensors quadratic in $\nabla \bar{g}$ with coefficients quadratic in \bar{g} and tensors that are quadratic in \bar{g} .

Then

$$A_{\mu\nu;\alpha}^{\alpha} - A_{\nu\alpha;\mu}^{\alpha} \approx -\frac{1}{2} \bar{g}^{\alpha\beta} \bar{g}_{\mu\nu;\alpha\beta} + \frac{1}{2} \bar{g}^{\alpha\beta} [\bar{g}_{\mu\beta;\nu\alpha} + \bar{g}_{\nu\alpha;\beta\mu} - \bar{g}_{\beta\alpha;\nu\mu}]. \quad (4)$$

Fact: Up to \approx , the two terms on the r.h.s of (4) are geometric relative to the background metric g .

1. $\bar{g}^{\alpha\beta}\bar{g}_{\mu\nu;\alpha\beta} = \langle \bar{g}, \nabla^2 \bar{g} \rangle_{\bar{g}}$
2. The 2nd term is related to the *tension field* H of the identity map $Id : (W, \bar{g}) \rightarrow (W, g)$. Recall that the tension field of a smooth map $f : (M, \bar{g}) \rightarrow (N, g)$ is a smooth vector field $\tau = \tau^A \partial_A$ along f given by

$$\tau^A = \Delta_{\bar{g}} f^A + \Gamma_{BC}^A \bar{g}^{\alpha\beta} f_{,\alpha}^B f_{,\beta}^C.$$

Taking (N, h) to be (W, g) and $f = Id$, one has

$$H^\alpha = \Delta_{\bar{g}} x^\alpha + \bar{g}^{\beta\delta} \Gamma_{\beta\delta}^\alpha = \bar{g}^{\beta\delta} (\Gamma_{\beta\delta}^\alpha - \bar{\Gamma}_{\beta\delta}^\alpha) = -\bar{g}^{\beta\delta} A_{\beta\delta}^\alpha.$$

One checks that

$$(L_H \bar{g})_{\mu\nu} = \bar{\nabla}_{\partial_\mu} H_\nu + \bar{\nabla}_{\partial_\nu} H_\mu \approx -\bar{g}^{\alpha\beta} [\bar{g}_{\mu\beta;\nu\alpha} + \bar{g}_{\nu\alpha;\beta\mu} - \bar{g}_{\beta\alpha;\nu\mu}]$$

where $L_H \bar{g}$ is the Lie derivative of \bar{g} along H .

Define

$$\text{Ric}^H(\bar{g}) := \text{Ric}(\bar{g}) + \frac{1}{2}L_H\bar{g} = -\frac{1}{2}\langle \bar{g}, \nabla^2(\bar{g}) \rangle_{\bar{g}} + F(\nabla\bar{g}, \bar{g}) \quad (5)$$

where F is a linear combination of $\bar{\nabla}\bar{g} * \nabla\bar{g} * \bar{g} * \bar{g}$ and $\bar{g} * \bar{g}$ with coefficients depending on the curvature of g .

Then

- (a) $\text{Ric}^H(\bar{g})$ is quasi-linear elliptic of the unknown $\{\bar{g}_{\mu\nu}\}$ if \bar{g} is sought to be Riemannian
- (b) $\text{Ric}^H(\bar{g})$ is quasi-linear hyperbolic of the unknown $\{\bar{g}_{\mu\nu}\}$ if \bar{g} is sought to be Lorentzian.

For instance, (a) explains why one often considers the modified evolution equation $\partial_t\bar{g} = -2\text{Ric}(\bar{g}) - L_H\bar{g}$ in the study of the Ricci flow $\partial_t\bar{g} = -2\text{Ric}(\bar{g})$ when \bar{g} is Riemannian.

We go back to the initial data setting: Let $\bar{M} = M \times \mathbb{R}^1$. Let $g = -dt^2 + \sigma$ be a background metric on \bar{M} .

We seek a Lorentz metric tensor \bar{g} on some open neighborhood W of $M \times \{0\}$ in \bar{M} such that

$$\text{Ric}^H(\bar{g}) = 0 \text{ on } W, \quad \bar{g}|_{TM} = \sigma \text{ and } \text{III} = K \quad (6)$$

Here we identify M with $M \times \{0\}$.

We have seen that the PDE system $\text{Ric}^H(\bar{g}) = 0$ is quasi-linear hyperbolic in $\{\bar{g}_{\mu\nu}\}$. To solve it, one needs to specify initial conditions

$$\bar{g}_{ij}|_{t=0}, \bar{g}_{0i}|_{t=0}, \bar{g}_{00}|_{t=0}, \partial_t \bar{g}_{ij}|_{t=0}, \partial_t \bar{g}_{0i}|_{t=0}, \partial_t \bar{g}_{00}|_{t=0}.$$

Here $i, j \in \{1, \dots, n\}$ and $\partial_0 = \partial_t$.

It is clear that $\bar{g}|_{TM} = \sigma$ translates into $\bar{g}_{ij}|_{t=0} = \sigma_{ij}$.

To have a clean formula relating $\partial_t \bar{g}_{00}|_{t=0}$ to III, it is natural to require $\bar{g}_{0i}|_{t=0} = 0$ and $\bar{g}_{00}|_{t=0} = -1$, that is

$$\partial_t \perp_{\bar{g}} M \text{ and } \langle \partial_t, \partial_t \rangle_{\bar{g}} = -1 \text{ at } t = 0.$$

With this specification of $\bar{g}|_{t=0}$, the condition III = K naturally translates into

$$\partial_t \bar{g}_{ij}|_{t=0} = 2K_{ij}.$$

It remains to specify $\partial_t \bar{g}_{0i}|_{t=0}$ and $\partial_t \bar{g}_{00}|_{t=0}$.

Claim: When $\bar{g}_{ij}|_{t=0}$, $\bar{g}_{0i}|_{t=0}$, $\bar{g}_{00}|_{t=0}$, $\partial_t \bar{g}_{ij}|_{t=0}$ are known, prescribing $\partial_t \bar{g}_{0\alpha}|_{t=0}$ is equivalent to prescribing H^β (check!)

Therefore, one considers the following initial value problem for the quasi-linear hyperbolic system

$$\left\{ \begin{array}{l} \text{Ric}^H(\bar{g}) = 0 \\ \bar{g}_{ij}|_{t=0} = \sigma_{ij} \\ \bar{g}_{0j}|_{t=0} = 0 \\ \bar{g}_{00}|_{t=0} = -1 \\ \partial_t \bar{g}_{ij}|_{t=0} = 2K_{ij} \\ H^\mu|_{t=0} = 0 \end{array} \right. \quad (7)$$

which has a solution $\{\bar{g}_{\mu\nu}\}$ on a small open set W containing $M \times \{0\}$ in $M \times \mathbb{R}^1$.

Claim: A solution (W, \bar{g}) to (7) satisfies $H^\mu = 0$.

$$0 = \text{Ric}^H(\bar{g}) \Rightarrow \begin{cases} 0 = R(\bar{g}) + \text{div}_{\bar{g}} H \\ 0 = \frac{1}{2} dR(\bar{g}) + \frac{1}{2} [d(\text{div}_{\bar{g}} H) + \square_{\bar{g}} H + \text{Ric}(\bar{g})(H, \cdot)] \end{cases}$$

Therefore, $\text{Ric}^H(\bar{g}) = 0 \Rightarrow \square_{\bar{g}} H + \text{Ric}(\bar{g})(H, \cdot) = 0$.

We do know $H^\mu|_{t=0} = 0$. On the other hands,

Vacuum Einstein Constraint Equation on $M \Rightarrow G_{0\mu}(\bar{g})|_{t=0} = 0$.

This, together with $\text{Ric}(\bar{g}) + \frac{1}{2} L_H \bar{g} = 0$ and $H^\mu|_{t=0} = 0$, implies

$$\partial_t H^\mu|_{t=0} = 0.$$

Therefore, H satisfies the linear hyperbolic system

$\square_{\bar{g}} H + \text{Ric}(\bar{g})(H, \cdot) = 0$, with $H|_{t=0} = 0$ and $L_{\partial_t} H|_{t=0} = 0$.

Therefore, $H = 0$ identically.

Theorem (Choquet-Bruhat and Geroch) (smooth version)

For a smooth vacuum initial data set (M, σ, K) , there exists a unique (up to isometry) spacetime (\bar{M}, \bar{g}) satisfying

1. $\text{Ric}(\bar{g}) = 0$ and \exists an isometric embedding $\iota : (M, \sigma) \rightarrow (\bar{M}, \bar{g})$ such that $\iota^*(\text{III}) = K$
2. $\iota(M)$ is a Cauchy hypersurface in (\bar{M}, \bar{g})
3. If (\bar{M}', \bar{g}') satisfies the above two conditions, then (\bar{M}', \bar{g}') is isometric to an open set of (\bar{M}, \bar{g}) .
4. (\bar{M}, \bar{g}) is inextendible in the class of smooth globally hyperbolic spacetimes.

(\bar{M}, \bar{g}) is often referred as the *maximal Cauchy development* of (M, σ, K) .

The theorem asserts that the maximal Cauchy development of (M, σ, K) is inextendible in the class of *globally hyperbolic* spacetimes.

There are examples where (\bar{M}, \bar{g}) can be extended in more than one way to strictly larger vacuum spacetimes that fail to be globally hyperbolic. These extended spacetimes can *not* be “predicted” by the given initial data set.

Strong Cosmic Censorship(simple version)

For a generic asymptotically flat vacuum initial data set, its maximal Cauchy development is inextendible in the class of spacetimes.

Let (M, σ, K) be a vacuum initial data set containing a bounded region Ω such that $\Sigma = \partial\Omega$ satisfying $\text{tr}_\Sigma K \pm H_\Sigma < 0$ where H_Σ is the mean curvature of Σ . By Penrose's Singularity Theorem, the maximal Cauchy development of (M, σ, K) is necessarily singular, i.e. it is *future null geodesically incomplete*.

Weak Cosmic Censorship (simple version)

For a generic asymptotically flat initial data set, its maximal Cauchy development has a “good” future null infinity.

A usual way of defining “future null infinity”, \mathcal{J}^+ , is through conformal compactification. Once one makes sense of \mathcal{J}^+ , the Black hole region \mathcal{B} is naturally defined to be the spacetime complement of $J^-(\mathcal{J}^+)$. The event horizon of \mathcal{B} is just $\partial\mathcal{B} = \partial J^-(\mathcal{J}^+)$.

Therefore, though singularity can occur in a maximal Cauchy development, Weak Cosmic Censorship asserts that, for a generic initial data set, no singularities can be observed at \mathcal{I}^+ , that is to say singularities are contained in a black hole \mathcal{B} .

The strong and weak cosmic censorships are logically independent.