Cauchy Problem and Cosmic Censorship

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An *initial data set* for the Vacuum Einstein Euation consists of a connected, n-dimensional manifold M together with a Riemannian metric σ and a symmetric $(0,2)$ tensor field K such that (M,σ,K) satisfies the Vacuum Einstein Constraint Equations:

$$\left\{\begin{array}{rcl} R_\sigma - |K|^2_\sigma + (\mathrm{tr}_\sigma K)^2 & = & 0 \\ \mathrm{div}_\sigma K - d(\mathrm{tr}_\sigma K) & = & 0. \end{array}\right. \tag{1}$$

Remark: σ and K are assumed to be smooth in this lecture.

Question: Does there exist a *globally hyperbolic* spacetime $(\bar{M}^{n+1}, \bar{g})$ satisfying

(1) the Vacuum Einstein Equation $\mathrm{Ric}(\bar{g}) = 0$

(2) there exists an isometric embedding

$$\iota : (M^n, \sigma) \longrightarrow (\bar{M}^{n+1}, \bar{g})$$

with $\iota^*(\mathbb{II}) = K$, where $\mathbb{II}$ is the second fundamental form of $\iota(\Sigma)$ in $(\bar{M}^{n+1}, \bar{g})$ w.r.t. the future timelike unit normal

(3) $\iota(M)$ is a Cauchy hypersurface in $(\bar{M}^{n+1}, \bar{g})$?

Recall that the domain of dependence $D(A)$ of an achronal set A in an arbitrary spacetime $\bar{M}$ satisfies the basic property that

$$\mathrm{int}(D(A)) \text{ is a globally hyperbolic open set in } \bar{M}.$$

Moreover, if A is an acausal C^0 hypersurface, then $int(A)$ is an open set itself.

Suppose one already has a spacetime $(\bar{M}^{n+1}, \bar{g})$ satisfying conditions (1) and (2) above and suppose $i(M)$ is acausal in $(\bar{M}^{n+1}, \bar{g})$. Replacing $\bar{M}^{n+1}$ by $D(\iota(M))$, one obtains a spacetime $(D(\iota(M), \bar{g})$ satisfying conditions (1)–(3).

When is a spacelike hypersurface an achronal/acausal set?

Lemma Let M be a spacelike hypersurface in a spacetime $\bar{M}$. If $\bar{M} \setminus M$ is disconnected, then M is achronal.

Lemma An acronal, spacelike hypersurface M is necessarily acausal.

Therefore, given a vacuum initial data set (M, σ, K), if one can find a spacetime metric $\bar{g}$ on an open set U, containing $M \times \{0\}$, in the product manifold $M \times \mathbb{R}^1$ such that

- $\mathrm{Ric}(\bar{g}) = 0$ on U
- $\iota : M \to M \times \{0\}$ satisfies $\iota^*(\bar{g}) = \sigma$ and $\iota^*(\mathbb{II}) = K$,

then $(D(\iota(M)), \bar{g})$ is a globally hyperbolic spacetime satisfying (1)-(3).

Analyzing the Ricci Curvature

Let $\bar{g}$ and g be two arbitrary metrics (Riemannian or Lorentzian) on an open set W. We treat g as a background metric. Let $\bar{\nabla}$, ∇ be the covariant differentiation associated to $\bar{g}$, g respectively.

It is a basic fact that $A = \bar{\nabla} - \nabla$ is a tensor on W. In any local coordinate chart $\{x^\mu\}$ on W, one has

$$A^{\delta}_{\mu\nu}\partial_\delta := \bar{\nabla}_{\partial_\mu}\partial_\nu - \nabla_{\partial_\mu}\partial_\nu$$

where

$$A^{\delta}_{\mu\nu} = \frac{1}{2}\bar{g}^{\delta\gamma}\left(\bar{g}_{\mu\gamma;\nu} + \bar{g}_{\gamma\nu;\mu} - \bar{g}_{\mu\nu;\gamma}\right) \tag{2}$$

where " ; " denotes the covariant differentiation w.r.t the background metric g.

In terms of A, direct calculation gives

$$\mathrm{Ric}(\bar{g})_{\mu\nu} - \mathrm{Ric}(g)_{\mu\nu} = A^{\alpha}_{\mu\nu;\alpha} - A^{\alpha}_{\nu\alpha;\mu} + A^{\delta}_{\mu\nu} A^{\alpha}_{\alpha\delta} - A^{\alpha}_{\mu\delta} A^{\delta}_{\alpha\nu}. \quad (3)$$

In what follows, we write $h_1 \approx h_2$ to suggest that $h_1 - h_2$ is a sum of tensors quadratic in $\nabla \bar{g}$ with coefficients quadratic in $\bar{g}$ and tensors that are quadratic in $\bar{g}$.

Then

$$A^{\alpha}_{\mu\nu;\alpha} - A^{\alpha}_{\nu\alpha;\mu} \approx -\frac{1}{2} \bar{g}^{\alpha\beta} \bar{g}_{\mu\nu;\alpha\beta} + \frac{1}{2} \bar{g}^{\alpha\beta} \left[\bar{g}_{\mu\beta;\nu\alpha} + \bar{g}_{\nu\alpha;\beta\mu} - \bar{g}_{\beta\alpha;\nu\mu} \right]. \quad (4)$$

Fact: Up to $\approx$, the two terms on the r.h.s of (4) are geometric relative to the background metric g.

1. $\bar{g}^{\alpha\beta}\bar{g}_{\mu\nu;\alpha\beta} = \langle \bar{g}, \nabla^2 \bar{g} \rangle_{\bar{g}}$

2. The 2nd term is related to the *tension field H* of the identity map $Id : (W, \bar{g}) \to (W, g)$. Recall that the tension field of a smooth map $f : (M, \bar{g}) \to (N, g)$ is a smooth vector field $\tau = \tau^A \partial_A$ *along f* given by

$$\tau^A = \Delta_{\bar{g}} f^A + \Gamma^A_{BC} \bar{g}^{\alpha\beta} f^B_{,\alpha} f^C_{,\beta}.$$

Taking (N, h) to be (W, g) and $f = Id$, one has

$$H^\alpha = \Delta_{\bar{g}} x^\alpha + \bar{g}^{\beta\delta}\Gamma^\alpha_{\beta\delta} = \bar{g}^{\beta\delta}(\Gamma^\alpha_{\beta\delta} - \bar{\Gamma}^\alpha_{\beta\delta}) = -\bar{g}^{\beta\delta} A^\alpha_{\beta\delta}.$$

One checks that

$$(L_H \bar{g})_{\mu\nu} = \bar{\nabla}_{\partial_\mu} H_\nu + \bar{\nabla}_{\partial_\nu} H_\mu \approx -\bar{g}^{\alpha\beta}\left[\bar{g}_{\mu\beta;\nu\alpha} + \bar{g}_{\nu\alpha;\beta\mu} - \bar{g}_{\beta\alpha;\nu\mu}\right]$$

where $L_H \bar{g}$ is the Lie derivative of $\bar{g}$ along H.

Define

$$\mathrm{Ric}^H(\bar{g}) := \mathrm{Ric}(\bar{g}) + \frac{1}{2}L_H\bar{g} = -\frac{1}{2}\langle \bar{g}, \nabla^2(\bar{g})\rangle_{\bar{g}} + F(\nabla\bar{g}, \bar{g}) \quad (5)$$

where F is a linear combination of $\overline{\nabla}\bar{g} * \nabla\bar{g} * \bar{g} * \bar{g}$ and $\bar{g} * \bar{g}$ with coefficients depending on the curvature of g.

Then

(a) $\mathrm{Ric}^H(\bar{g})$ is quasi-linear elliptic of the unknown $\{\bar{g}_{\mu\nu}\}$ if $\bar{g}$ is sought to be Riemannian

(b) $\mathrm{Ric}^H(\bar{g})$ is quasi-linear hyperbolic of the unknown $\{\bar{g}_{\mu\nu}\}$ if $\bar{g}$ is sought to be Lorentzian.

For instance, (a) explains why one often considers the modified evolution equation $\partial_t\bar{g} = -2\mathrm{Ric}(\bar{g}) - L_H\bar{g}$ in the study of the Ricci flow $\partial_t\bar{g} = -2\mathrm{Ric}(\bar{g})$ when $\bar{g}$ is Riemannian.

We go back to the initial data setting: Let $\bar{M} = M \times \mathbb{R}^1$. Let $g = -dt^2 + \sigma$ be a background metric on $\bar{M}$.

We seek a Lorentz metric tensor $\bar{g}$ on some open neighborhood W of $M \times \{0\}$ in $\bar{M}$ such that

$$\mathrm{Ric}^H(\bar{g}) = 0 \text{ on } W, \ \bar{g}|_{TM} = \sigma \text{ and } \mathbb{II} = K \tag{6}$$

Here we identify M with $M \times \{0\}$.

We have seen that the PDE system $\mathrm{Ric}^H(\bar{g}) = 0$ is quasi-linear hyperbolic in $\{\bar{g}_{\mu\nu}\}$. To solve it, one needs to specify initial conditions

$$\bar{g}_{ij}|_{t=0}, \ \bar{g}_{0i}|_{t=0}, \ \bar{g}_{00}|_{t=0}, \ \partial_t \bar{g}_{ij}|_{t=0},, \ \partial_t \bar{g}_{0i}|_{t=0}, \ \partial_t \bar{g}_{00}|_{t=0}.$$

Here $i, j \in \{1, \dots, n\}$ and $\partial_0 = \partial_t$.

It is clear that $\bar{g}|_{TM} = \sigma$ translates into $\bar{g}_{ij}|_{t=0} = \sigma_{ij}$.

To have a clean formula relating $\partial_t \bar{g}_{00}|_{t=0}$ to $\mathbb{II}$, it is natural to require $\bar{g}_{0i}|_{t=0} = 0$ and $\bar{g}_{00}|_{t=0} = -1$, that is

$$\partial_t \perp_{\bar{g}} M \text{ and } \langle \partial_t, \partial_t \rangle_{\bar{g}} = -1 \text{ at } t = 0.$$

With this specification of $\bar{g}|_{t=0}$, the condition $\mathbb{II} = K$ naturally translates into

$$\partial_t \bar{g}_{ij}|_{t=0} = 2K_{ij}.$$

It remains to specify $\partial_t \bar{g}_{0i}|_{t=0}$ and $\partial_t \bar{g}_{00}|_{t=0}$.

Claim: When $\bar{g}_{ij}|_{t=0}, \ \bar{g}_{0i}|_{t=0}, \ \bar{g}_{00}|_{t=0}, \ \partial_t \bar{g}_{ij}|_{t=0}$ are known, prescribing $\partial_t \bar{g}_{0\alpha}|_{t=0}$ is equivalent to prescribing H^β (check!)

Therefore, one considers the following initial value problem for the quasi-linear hyperbolic system

$$\begin{cases} \mathrm{Ric}^H(\bar{g}) & = & 0 \\ \bar{g}_{ij}|_{t=0} & = & \sigma_{ij} \\ \bar{g}_{0j}|_{t=0} & = & 0 \\ \bar{g}_{00}|_{t=0} & = & -1 \\ \partial_t \bar{g}_{ij}|_{t=0} & = & 2K_{ij} \\ H^\mu|_{t=0} & = & 0 \end{cases} \tag{7}$$

which has a solution $\{\bar{g}_{\mu\nu}\}$ on a small open set W containing $M \times \{0\}$ in $M \times \mathbb{R}^1$.

Claim: A solution $(W, \bar{g})$ to (7) satisfies $H^\mu = 0$.

$$0 = \mathrm{Ric}^H(\bar{g}) \Rightarrow \begin{cases} 0 = R(\bar{g}) + \mathrm{div}_{\bar{g}} H \\ 0 = \frac{1}{2} dR(\bar{g}) + \frac{1}{2}[d(\mathrm{div}_{\bar{g}} H) + \Box_{\bar{g}} H + \mathrm{Ric}(\bar{g})(H, \cdot)] \end{cases}$$

Therefore, $\mathrm{Ric}^H(\bar{g}) = 0 \Rightarrow \Box_{\bar{g}} H + \mathrm{Ric}(\bar{g})(H, \cdot) = 0$.

We do know $H^\mu|_{t=0} = 0$. On the other hands,

$$\text{Vacuum Einstein Constraint Equation on } M \Rightarrow G_{0\mu}(\bar{g})|_{t=0} = 0.$$

This, together with $\mathrm{Ric}(\bar{g}) + \frac{1}{2} L_H \bar{g} = 0$ and $H^\mu|_{t=0} = 0$, implies

$$\partial_t H^\mu|_{t=0} = 0.$$

Therefore, H satisfies the linear hyperbolic system $\Box_{\bar{g}} H + \mathrm{Ric}(\bar{g})(H, \cdot) = 0$, with $H|_{t=0} = 0$ and $L_{\partial_t} H|_{t=0} = 0$. Therefore, $H = 0$ identically.

Theorem (Choquet-Bruhat and Geroch) (smooth version)

For a smooth vacuum initial data set (M, σ, K), there exists a unique (up to isometry) spacetime $(\bar{M}, \bar{g})$ satisfying

1. $\mathrm{Ric}(\bar{g}) = 0$ and $\exists$ an isometric embedding $\iota : (M, \sigma) \to (\bar{M}, \bar{g})$ such that $\iota^*(\mathbb{II}) = K$
2. $\iota(M)$ is a Cauchy hypersurface in $(\bar{M}, \bar{g})$
3. If $(\bar{M}', \bar{g}')$ satisfies the above two conditions, then $(\bar{M}', \bar{g}')$ is isometric to an open set of $(\bar{M}, \bar{g})$.
4. $(\bar{M}, \bar{g})$ is inextendible in the class of smooth globally hyerpbolic spacetimes.

$(\bar{M}, \bar{g})$ is often referred as the *maximal Cauchy development* of (M, σ, K).

The theorem asserts that the maximal Cauchy development of (M, σ, K) is inextendible in the class of *globally hyperbolic* spacetimes.

There are examples where $(\bar{M}, \bar{g})$ can be extended in more than one way to strictly larger vacuum spacetimes that fail to be globally hyperbolic. These extended spacetimes can *not* be "predicted" by the given initial data set.

Strong Cosmic Censorship(simple version)
For a generic asymptotically flat vacuum initial data set, its maximal Cauchy development is inextendible in the class of spacetimes.

Let (M, σ, K) be a vacuum initial data set containing a bounded region Ω such that $\Sigma = \partial\Omega$ satisfying $\mathrm{tr}_\Sigma K \pm H_\Sigma < 0$ where H_Σ is the mean curvature of Σ. By Penrose's Singularity Theorem, the maximal Cauchy development of (M, σ, K) is necessarily singular, i.e. it is *future null geodesically incomplete*.

Weak Cosmic Censorship (simple version)

For a generic asymptotically flat initial data set, its maximal Cauchy development has a "good" future null infinity.

A usual way of defining "future null infinity", $\mathcal{J}^+$, is through conformal compactification. Once one makes sense of $\mathcal{J}^+$, the Black hole region $\mathcal{B}$ is naturally defined to be the spacetime complement of $J^-(\mathcal{J}^+)$. The event horizon of $\mathcal{B}$ is just $\partial\mathcal{B} = \partial J^-(\mathcal{J}^+)$.

Therefore, though singularity can occur in a maximal Cauchy development, Weak Cosmic Censorship asserts that, for a generic initial data set, no singularities can be observed at $\mathcal{J}^+$, that is to say singularities are contained in a black hole $\mathcal{B}$.

The strong and weak cosmic censorships are logically independent.