PROJECTS FOR MSRI SUMMER SCHOOL IN COMMUTATIVE ALGEBRA

These are outlines of various groups projects. We will ask you to list a few of the projects that you would be interested in working on. Feel free to ask any of the organizers or TAs for further information about any of the projects. In addition, we will spend some time on Wednesday answering any questions you might have about any of the projects.

1. Updating Seidenberg's Constructive Approach to Computation

The goal of this project is to update Seidenberg's paper (A. Seidenberg, Constructions in algebra, *Trans. Amer. Math. Soc.* **197** (1974), 273–313) by incorporating Groebner basis results, keeping track of the changes on upper bounds in the computations while doing reductions, moving the 96 points around into a more streamlined paper. As this problem involves 96 problems on 40 pages, perhaps subgroups could be formed tackling parts: upper bounds on colons, on syzygies, transcendence degrees, Kronecker's trick of multivariate factorization, primary decomposition (degrees and the number of generators for the components and for the associated primes), radicals...

Furthermore, there is no reason to stick only to Seidenberg's topics.

2. The Seidenberg-Stolzenberg "algorithm" to compute Integral Closure

Implement the Seidenberg-Stolzenberg "algorithm" for computing integral closure of rings, say over characteristic 0. The big point is to construct an (R_1)ification of a ring first. One needs the Jacobian ideal first to see how and if the ring satisfies Serre's condition (R_1). Take a prime ideal P of height 1 such that R_P is not regular. Then for any $a, b \in P$ that locally are part of a minimal generating set of P, there exists a unit $u \in R$ such that $a/(a + ub)$ is integral over R_P and is not in R_P. Write an algorithm that makes this u computable and that determines integrality. Then find some R-multiple of this element that is integral over R and is not in R. Then repeat (R_1)ification on the larger ring obtained from R by adjoining this integral element.

For background, see:

- G. Stolzenberg, Constructive normalization of an algebraic variety. *Bull. Amer. Math. Soc.* **74** (1968), 595–599.
- A. Seidenberg, Construction of the integral closure of a finite integral domain, *Rendiconti Sem. matematico e fisico, Milano* **40** (1970), 101–120.
- A. Seidenberg, Construction of the integral closure of a finite integral domain, II, *Proc. Amer. Math. Soc.* **52** (1975), 368–372.
- Or, for a more modern rendition, see Section 15.1 in C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.

Compare the efficiency of your algorithm with the established ones.

3. CI-models and binomial ideals

We argued that every CI-model where all marginalizations use only one variable, can be given as a binomial ideal after a linear change of variables and we saw that the same transformation does not work for marginalizations of 2 or more variables. This leads to two questions:

(1) Are there other CI-models that are not saturated where some other linear change of variables gives a binomial ideal? What classes of models have this feature? How do we then interpret the primary decomposition?
(2) Choose the simplest CI-model with at least 2 variables marginalized and no obvious linear change of variables to a binomial ideals and explore its primary decomposition. What sorts of patterns can you find and prove?

4. The Primary Components of the Ideals from [ST]

(1) The ideals for which [ST] give the minimal primes, are, in general, not radical. Classify the embedded primes for these ideals. In particular, do they correspond to admissible sets and in what way. Cases where the maximal ideal is embedded is particularly interesting.
(2) What are the heights of the primes associated to the ideals studied by [ST].
(3) When $n = 3$, [ST] give the minimal components for these ideals. Prove these ideals are in fact radical. Note that it is enough to prove there are no embedded components.
(4) Find a nice description of the radicals of the ideals studied by [ST] (when $n \geq 4$, of course).

5. Extending Cellular Decompositions

An ideal $I \subset S$ is primary (cellular) if every element (monomial) of S/I is either a nonzerodivisor or nilpotent. There is a very simple algorithm for cellular decomposition using Proposition 7.2 of Eisenbud and Sturmfels. Design an algorithm for primary decomposition of general ideals along the same lines.

Try to implement your algorithm in Macaulay2. What is the Macaulay2-function "primaryDecomposition" doing instead? Identify a class of ideals where your your algorithm works and it is faster than "primaryDecomposition".

6. Reducedness of certain CI-models

The ideals studied by [HHH$^+$10] and in [GSS05] the Bayesian networks for 4 binary random variables and many for 5 binary random variables are radical. However, those studied by [ST] and those networks that are not binary are not. Is there another large class of CI-models that are radical? Can we classify the embedded primes in these cases? What might the interpretation of the embedded primes be?

7. Minors of Hypermatrices

The ideals studied by [ST] are 2×2 minors of hypermatrices. The 2×2 minors are straightforward to define, but larger minors are more challenging. The results of [ST] for 2×2 minors are similar to those for 2×2 minors of matrices. Are there analogous theorems for larger minors of hypermatrices? Where for a d-minor of a hypermatrix we use the

definitions (2.3 and 2.4) given in Alessandra Bernardi's paper [Ber08]. Of course, then all of the questions asked about the ideals studied by [ST] can be asked about the larger minors.

8. Experiments with Conditional Independence Models

Using Macualay2 one can compute the primary decomposition for I_{global} with less than or equal to 5 vertices. What happens for less than or equal to 6 vertices for I_{pairwise} or I_{local}?

9. Free Resolutions and Algebraic Statistics

For even the simplest CI-models and graphical models, is there a nice description of the free resolution? Does the resolution data seem to have interesting connections to the model?

10. The Boij-Söderberg theory of monomial ideals

This project would be especially appropriate for someone with combinatorial interests. A good general reference on Stanley–Reisner theory is given in [BH93, Chapter 5]. See also [MS05, Chapter 1].

Fix a simplicial complex Δ and let I_Δ be the corresponding Stanley–Reisner ideal. Thus, $I_\Delta \subseteq S$ is a squarefree monomial ideal. Consider the Boij-Söderberg decompositions

$$\beta(S/I_\Delta) = \sum a_d \pi_d \qquad \text{and} \qquad \beta(I_\Delta) = \sum b_d \pi_d.$$

The coefficients $\{a_d\}$ and $\{b_d\}$ provide numerical invariants of the ideal I_Δ. Do these coefficients have any relationship to properties of the simplicial complex Δ?

Example 10.1. *Let $S = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$. Let*

$$I := (x_1, x_2, x_3) \cap (x_4, x_5).$$

The corresponding simplical complex is the union of a triangle and a line segment. We have the Boij–Söderberg decomposition:

$$\begin{aligned}\beta(S/I) &= 30 \cdot \pi_{(0,2,3,4,5)} + 10 \cdot \pi_{(0,2,3,4)} + 2 \cdot \pi_{(0,2,3)} \\ &= \begin{pmatrix} \frac{1}{4} & - & - & - & - \\ - & \frac{5}{2} & 5 & \frac{15}{4} & 1 \end{pmatrix} + \begin{pmatrix} \frac{5}{12} & - & - & - \\ - & \frac{5}{2} & \frac{10}{3} & \frac{5}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & - & - \\ - & 1 & \frac{2}{3} \end{pmatrix}.\end{aligned}$$

How are the 3 pure summands on the right related to the properties of I (and the corresponding simplicial complex)?

Cases to consider:

10.1. Product of ideals.

Let $S = k[x_1, \dots, x_n, y_1, \dots, y_m]$ and set $I_\Delta = (x_1, \dots, x_n) \cap (y_1, \dots, y_m)$. Some possible questions to start with:

(1) What is the corresponding simplicial complex Δ in this case?
(2) Let $n = m$. Can you work a closed formula for either the a_d or the b_d in this case?

10.2. **Triangulations of manifolds.** Fix a 2-dimensional manifold (orientable or unorientable) and fix some triangulation Δ of the surface. Consider the corresponding ideal I_Δ.

(1) What happens as you vary the triangulation? (Think about the famous formula $V - E + F$, which doesn't depend on the choice of triangulation. Can you find any sort of analogous stability statement among the a_d or b_d?)
(2) Consider $\mathbb{RP}^2$ with the triangulation described in [BH93, Figure 5.8]. We see that $\beta(S/I_\Delta)$ depends on the characteristic of the ground field. Compute the Boij–Söderberg decomposition in charactersitic 0 and characteristic 2.

10.3. **Alexander Duality.** How does BS decomposition of S/I_Δ compare with BS decomposition of S/I_Δ^{AD}? See Miller–Sturmfels' book [MS05, Chapter 5] for details on Alexander Duality and many possible examples to consider.

11. Boij–Söderberg Decompositions of Binomial ideals

Binomial ideals are a central subject in combinatorial commutative algebra. Toric ideals are closely connected with toric varieties and semigroups. See [MS05, Chapter 9] for an introduction. In this project, the goal is to analyze connections between the Boij–Söderberg decompsition of a binomial ideal and the algebraic/combinatorial properties of the ideal.

11.1. **Toric Ideals.** Given a matrix $A \in \mathbb{Z}^{d\times n}$, set

$$I_A := \langle x^u - x^v \mid Au = Av \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

Write

$$\beta(S/I_A) = \sum a_d \pi_d \qquad \text{and} \qquad \beta(I_A) = \sum b_d \pi_d.$$

(1) Do these coefficients have combinatorial meaning for A?
(2) Can some multiple of these summands be realized by toric ideals/modules?

Example 11.1. *If* $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ *then the Boij–Söderberg decomposition for* $\mathbb{C}[x]/I_A$ *is*

$$\begin{aligned}\beta(\mathbb{C}[x]/I_A) &= 4 \cdot \pi_{(0,2,4,5)} + 2 \cdot \pi_{(0,3,4,5)} + 8 \cdot \pi_{(0,3,4)} \\ &= \begin{pmatrix} \frac{3}{10} & - & - & - \\ - & 1 & - & - \\ - & - & \frac{3}{2} & \frac{4}{5} \end{pmatrix} + \begin{pmatrix} \frac{1}{30} & - & - & - \\ - & - & - & - \\ - & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} & - & - \\ - & - & - \\ - & \frac{8}{3} & 1 \end{pmatrix}.\end{aligned}$$

11.2. **Binomial primary decomposition.** This subproject is somewhat related to Project 13. Through the work of Eisenbud–Sturmfels and Dickenstein–Matusevich–Miller, binomial primary decomposition is explicitly understood. Also, mesoprimary decomposition is a related combinatorial notion that will appear in a forthcoming paper by Kahle–Miller. Is there interplay between Boij–Söderberg decomposition and binomial (meso)primary decomposition?

12. Boij–Söderberg Coefficients of Complete Intersections

There is no nontrivial family of examples where we understand the Boij–Söderberg decomposition in detail. The goal of this project is to produce the first such family of examples.

Consider a complete intersection of polynomials $f_1, \ldots, f_c$ where $\deg(f_i) = e_i$. Let $I = (f_1, \ldots, f_c)$ and $M = S/I$. Note that $\beta(S/I)$ only depends on the collection of positive integers $\{e_1, \ldots, e_c\}$. Write out the Boij-Söderberg decomposition of S/I:

$$\beta(S/I) = \sum a_d \pi_d.$$

There must be a formula yielding

$$(e_1, \ldots, e_c) \mapsto \sum_i a_i \pi_{d^i}.$$

(1) What are the d^i in terms of $e_1, \ldots, e_c$.
(2) What are the a_i?

It probably makes sense to start by trying to work out the formula when c is small (like $c = 2$ then $c = 3$).

13. Extended Notions of Multiplicity for Modules

Let M be a non-Cohen–Macaulay graded S-module. So we have $\mathrm{pdim}(M) := p > c := \mathrm{codim}(M)$. Consider the Boij-Söderberg decomposition of $\beta(M)$:

$$\beta(M) = \sum_d a_d \pi_d.$$

The general version of Boij–Söderberg's decomposition tells us that all pure diagrams arising in this sum have length between c and p. We may thus define D_i as the sum of all $a_d \pi_d$ where d has length i. We then obtain

$$\beta(M) = D_c + D_{c+1} + \cdots + D_p$$

where D_i is (up to scalar multiple) the Betti diagram of a Cohen–Macaulay module. Recall that $e(M)$ denotes the multiplicity of a module. Since the multiplicity can be computed entirely in terms of the Betti diagram, we may talk about the multiplicity of an abstract diagram.

Definition 13.1. $e_i^{BS}(M) := e(D_i)$.

This provides a notion of the "Cohen–Macaulay codimension i multiplicity" of a module M. What is the significance of this extended notion of multiplicity?

13.1. Depth 0 Cyclic Modules.

Let $M = S/I$ where I is a depth 0 ideal. In other words, we want that $\mathfrak{m} \in \mathrm{Ass}(M)$. How does $e_n^{\mathrm{BS}}(S/I)$ relate to the multiplicity of the $\mathfrak{m}$-primary embedded component of I?

Even more specifically, let J be some simple ideal and let $I = J \cap \mathfrak{m}^m$. Can you find a closed formula for $e_n^{\mathrm{BS}}(S/I)$ in this case? Start with some very simple examples like, like when J is principal.

References

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