Harmonic Functions on $\mathbb{R}^n$

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Introduction

We will use conformal changes of metric to deform the scalar curvature, and so we will want to understand the behavior of the Laplace operator Δ_g, as for $u > 0$:

$$R(u^{4/(n-2)}g) = -\tfrac{4(n-1)}{n-2}u^{-\frac{n+2}{n-2}}(\Delta_g u - \tfrac{n-2}{4(n-1)}R(g)u).$$

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Recall that the Hessian of a function on a Riemannian (or Lorentzian) manifold (M, g) is $\nabla(du)$, the components of which are denoted

$$u_{;ij} = \nabla(du)(\partial_i, \partial_j) = (\nabla_{\partial_j} du)(\partial_i) = u_{,ij} - du(\nabla_{\partial_j}\partial_i) = u_{,ij} - \Gamma^k_{ij} u_{,k}.$$

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On a Riemannian manifold, we write $\Delta_g u = g^{ij}u_{;ij}$, which can also be written $\Delta_g u = \mathrm{div}_g(\mathrm{grad}_g u)$.
We often write $\Delta = \Delta_{g_E}$

On a Lorentzian manifold, the trace of the Hessian gives the *wave operator* $\Box_g$.

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More generally, prescribing the scalar curvature of the conformal metric gives a semi-linear elliptic equation; prescribing that it vanish gives the linear equation $\Delta_g u - \frac{n-2}{4(n-1)} R(g) u = 0$.

So it is important to understand the behavior of operators of the form $(\Delta_g - f)$.

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Isolated gravitational system: If σ is compactly supported (or decays suitably at infinity), then how should we expect Φ to behave?

Spherically symmetric harmonic functions

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Now consider $\mathbb{R}^n$ with the Euclidean metric. If $u(x) = u(|x|)$ (slight abuse of notation), then with $r = |x|$, $g_E = dr^2 + r^{n-1} g_{\mathbb{S}^{n-1}}$, so

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So $\Delta u = 0$ is equivalent to $0 = \frac{d}{dr}\left(r^{n-1} \frac{du}{dr} \right)$.
Integrating we obtain (A and B are constants)

$$u(r) = \begin{cases} Ar + B\ , & n = 1 \\ A \log r + B\ , & n = 2 \\ Ar^{2-n} + B\ , & n > 2 \end{cases}$$

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As distributions,

$$\Delta\left(\tfrac{1}{2\pi}\log|x|\right) = \delta_0$$
$$\Delta\left(\tfrac{1}{(2-n)n\Omega_n}|x|^{2-n}\right) = \delta_0$$

$\Omega_n = \mathrm{vol}(\mathbb{B}^n(1))$ is the volume of the Euclidean unit ball. Note $n\Omega_n = \omega_{n-1}$ is the area of the unit sphere.

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Let $\Gamma(x, y) = \frac{1}{(2-n)n\Omega_n}|x-y|^{2-n}$ for $n > 2$, and $\Gamma(x, y) = \frac{1}{2\pi}\log|x-y|$ for $n = 2$.

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What this means is that for $u \in C_c^2(\mathbb{R}^n)$,

$$u(x) = \begin{cases} \int_{\mathbb{R}^2} \frac{1}{2\pi} \log|x-y| \Delta u(y)\, dy\ , & n = 2 \\ \int_{\mathbb{R}^n} \frac{1}{(2-n)n\Omega_n} |x-y|^{2-n} \Delta u(y)\, dy\ , & n > 2 \end{cases}$$

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Exercise

If $\Delta u = f$ compactly supported, $n > 2$, expand $|x-y|^{2-n}$ for large $|x|$ to obtain an expansion of u near infinity:

$$u(x) = \frac{A}{|x|^{n-2}} + \frac{x^i B_i}{|x|^n} + O(|x|^{-n}).$$

Find the constants A, B_i.

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$$u(a) = \frac{1}{n\Omega_n R^{n-1}} \int\limits_{|x-a|=R} u(x)\, d\sigma = \frac{1}{\Omega_n R^n} \int\limits_{B_R(a)} u(x)\, dx.$$

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Proof: Apply Green's identity $\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\, d\sigma$ with $v = |x-a|^{2-n}$ (in case $n > 2$, and analogously in case $n = 2$), with $\Omega = \{x : \epsilon < |x-a| < R\}$. Let $\epsilon \to 0^+$ and use continuity.

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If Ω bounded, u is harmonic on Ω and continuous on $\overline{\Omega}$, then the max and min of u are attained on $\partial\Omega$.

Dirichlet problem on a ball

Let $B_R = B_R(0)$. Let $P(x,y) = \frac{R^2 - |x|^2}{n\Omega_n R|x-y|^n}$. Then for $x \in B_R$, $\int_{y \in \partial B_R} P(x,y)\, d\sigma_y = 1$.

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Dirichlet problem

If $\varphi \in C(\partial B_R)$, then $u(x) = \int_{y\in\partial B_R} \varphi(y)P(x,y)\, d\sigma_y$ is harmonic on B_R, continuous on $\overline{B_R}$, and $u = \varphi$ on ∂B_R.

Poisson equation on a ball

For any $y \in B_R$, solve $\Delta h_y(x) = 0$ on B_R with $h_y(x) = -\Gamma(x, y)$ for $x \in \partial B_R$. Let $G(x, y) = \Gamma(x, y) + h_y(x)$.

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Applying Green's identity, and the distributional Laplacian of Γ (and hence of the *Green's function* G), we have

Poisson equation on ball

$$u(x) = \int_{\partial B_R} \varphi(y) P(x, y)\, d\sigma_y + \int_{B_R} G(x, y) f(y)\, dy$$

solves $\Delta u = f$ on B_R, $u = \varphi$ on ∂B_R.

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Hence $u = v$.

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$\frac{1}{R^n}((R + |x|)^n - (R - |x|)^n) u(0) = O(R^{-1})$.

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Let $R \to +\infty$.

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Harnack inequality

Ω connected domain, $K \subset \Omega$ compact. There exists $C > 0$ so that for all positive harmonic functions on Ω, for all $x, y \in K$,

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Idea: Use MVP. On unit ball B, the following estimate holds for $u \geq 0$:

$$u(x) = \frac{1}{n\Omega_n} \int_{\partial B} \frac{1 - |x|^2}{|x - y|^n} u(y)\, d\sigma_y \leq \frac{1 - |x|^2}{(1 - |x|^n)} u(0).$$

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Similarly,

$$u(x) = \frac{1}{n\Omega_n} \int_{\partial B} \frac{1 - |x|^2}{|x - y|^n} u(y)\, d\sigma_y \geq \frac{1 - |x|^2}{(1 + |x|)^n} u(0).$$

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$$u(x) = \begin{cases} b\log(|x|^{-1}) + v(x)\ , & n = 2 \\ b|x|^{2-n} + v(x)\ , & n > 2 \end{cases}$$

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$g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_E$ is the spatial Schwarzschild metric of mass m.

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$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to +\infty} \int\limits_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \nu_e^j \, d\sigma_e.$$

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In case $n = 3$, $m = \frac{1}{16\pi} \lim_{r \to +\infty} \int\limits_{|x|=r} \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j}) \nu_e^j \, d\sigma_e$.

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The flux integral limit provides the definition of mass/energy for more general asymptotically flat metrics.

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Inversion in the unit sphere:

$$x \mapsto x^* = |x|^{-2}x, \quad 0 \mapsto \infty, \quad \infty \mapsto 0.$$

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For functions u defined on $\Omega \subset \mathbb{R}^n \setminus \{0\}$, define $K[u]$ on Ω^* by

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Key exercise: u harmonic if and only if $K[u]$ is harmonic.

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$n > 2$. u is harmonic on $\mathbb{R}^n \setminus K$. u is harmonic at infinity if and only if $\lim_{|x| \to +\infty} u(x) = 0$.

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Lemma

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Proof: Exercise! Uses $u(x) = |x^*|^{n-2} K[u](x^*)$.

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The result is a function which is harmonic at infinity.

Example: $u(x) = (1 - |x|^{2-n})$ is harmonic and $u = 0$ on ∂B. The unique solution of the exterior Dirichlet problem with zero BC's is the zero function. $u(x)$ here is not harmonic at infinity, but $u - 1$ is.

Harmonic functions at ∞

Theorem

For $n > 2$, u harmonic on $\mathbb{R}^n \setminus K$, the following are equivalent.

- u is bounded near infinity
- u is bounded above or below near infinity
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Proof: Mostly easy. One real step involves using Bôcher's Theorem: if $u > 0$ near infinity, $K[u] > 0$ near the origin, so by Bôcher, there is a c so that $K[u](x) - c|x|^{2-n} = v(x)$ is harmonic near 0. So $K[v](x) = u(x) - c$ is harmonic near infinity.

Spherical harmonic expansion

If v harmonic at infinity,

$$|x^*|^{2-n} v(x) = K[v](x^*) = a_0 + a_i (x^*)^i + \cdots$$

by Taylor expansion of $K[v]$ about $x^* = 0$.

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Remark: The eigenfunctions of Δ_{g_0} on the unit sphere $(\mathbb{S}^n, g_0)$ are given by restrictions of homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$. The first eigenvalue $\lambda_0 = 0$ (constant functions), and the next eigenvalue is $\lambda_1 = n$, with eigenfunctions x^i, $i = 1, \ldots, n+1$ restricted to the sphere.

And now, for something completely different...

Asymptotically Euclidean Metrics and the Positive Mass Theorem

Justin Corvino

Lafayette College
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June 12, 2012

Asymptotically flat metrics

Definition

A complete Riemannian manifold (M^n, g), with an associated symmetric $(0,2)$ tensor K (or $\pi = K - (\mathrm{tr}_g K)g$) is called *asymptotically flat* (or *asymptotically Euclidean*) if there is a compact set $\mathcal{K} \subset M$ so that $M \setminus \mathcal{K}$ equals a disjoint union of asymptotic ends $\bigcup_{j=1}^{k} E_j$, where each asymptotic end E_j is diffeomorphic to $\mathbb{R}^n \setminus \{|x| \leq 1\}$, with asymptotically flat coordinates x^i in which the following decay estimates hold for multi-indices α and β:

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$$\begin{aligned} |\partial_x^\alpha (g_{ij} - \delta_{ij})(x)| &= O(|x|^{-|\alpha|-q}) \\ \left|\partial_x^\beta K_{ij}(x)\right| &= O(|x|^{-|\beta|-1-q}) \end{aligned}$$

for $|\alpha| \leq \ell + 1$, $|\beta| \leq \ell$, $q > \frac{n-2}{2}$ (so $q > 1/2$ in case $n = 3$).

Example: Schwarzschild

Schwarzschild $m > 0$

$(\mathbb{R}^n \setminus \{0\}, g_S)$ with
$g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_E$ is asymptotically flat with two ends if $m > 0$.
$q = n - 2$ in this case.

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Case $n = 3$: $(\mathbb{R}^3 \setminus \{|x| \leq -m/2\}, g_S)$.

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This metric is not complete.

Note: if we allow manifolds with boundary, we need to impose something on the boundary: compare Schwarzschild $m > 0$ and $m < 0$.

For simplicity, let's stick to the boundary-less case.

Example: harmonically flat metrics

Consider an asymptotically flat metric with an end E with asymptotically flat coordinates $x = (x^i)$, so that for $|x| > r_0$, $g = u^4 g_E$, $u \to 1$ at infinity, and $R(g) = 0$, i.e. $\Delta u = 0$.

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As we saw earlier, u admits an expansion:

$$u(x) = 1 + \frac{A}{|x|^{n-2}} + \frac{\beta \cdot x}{|x|^n} + \cdots$$

where A is a constant and β is a vector. Note that $q = n - 2$ here.

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Note also that derivatives fall off one order faster: $\frac{\partial |x|^{-1}}{\partial x^i} = -|x|^{-3} x^i$.

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Then we'll see $m = 2A$ (as in Schwarzschild), and if $A \neq 0$, we define $c = \beta/A$: a straightforward computation shows

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It makes sense then to identify c as the *center of mass*.

Expansion of scalar curvature

We have $g_{ij} - \delta_{ij} = O(|x|^{-q})$ (and same for inverse metric g^{ij}), $\partial g_{ij} = O(|x|^{-q-1})$ and $\partial^2 g_{ij} = O(|x|^{-q-2})$.

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$$\begin{aligned}
\Gamma_{ij,\ell}^k &= \frac{1}{2} g^{km}_{,\ell}(g_{im,j} + g_{mj,i} - g_{ij,m}) + \frac{1}{2} g^{km}(g_{im,j\ell} + g_{mj,i\ell} - g_{ij,m\ell}) \\
&= O(|x|^{-2q-2}) + \frac{1}{2}\delta^{km}(g_{im,j\ell} + g_{mj,i\ell} - g_{ij,m\ell}) \\
&+ \frac{1}{2}(g^{km} - \delta^{km})(g_{im,j\ell} + g_{mj,i\ell} - g_{ij,m\ell}) \\
&= \frac{1}{2}(g_{ik,j\ell} + g_{kj,i\ell} - g_{ij,k\ell}) + O(|x|^{-2q-2}).
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Note that the error term is in $L^1(E)$: for $q > (n-2)/2$, $2q+2 > n$.

The mass (energy) integral

Energy

Recall the mass (energy) integral is given by

$$\frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to+\infty} \int\limits_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e.$$

Note $\nu_e^j = \frac{x^j}{|x|}$.

In case $n = 3$, this is $\frac{1}{16\pi} \lim_{r\to+\infty} \int_{|x|=r} \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e$.

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Does the limit exist?

The mass (energy) integral

Well, by the divergence theorem,

$$\int_{|x|=r} - \int_{|x|=r_0} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e = \int_{r_0 \le |x| \le r} \sum_{i,j=1}^{n} (g_{ij,ij} - g_{ii,jj}) \, dx$$

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From the constraint $R(g) - \|K\|_g^2 + H^2 = 2\kappa\rho$, with $K_{ij} = O(|x|^{-q-1})$, we see $R(g) \in L^1(E)$ if and only if $\rho \in L^1(E)$, e.g. vacuum case $\rho = 0$.

The energy-momentum four-vector

Remarks

- The limit can be non-zero even if $R(g) = 0$ (vacuum), e.g. Schwarzschild.

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Now recall the second constraint equation $\mathrm{div}_g(\pi) = \kappa J$. Now

$$(\mathrm{div}_g \pi)_i = g^{jk}(\pi_{ij,k} - \Gamma^m_{ik}\pi_{mj} - \Gamma^m_{kj}\pi_{im}) = \mathrm{div}_e(\pi) + O(|x|^{-2q-2})$$

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So for $|J| \in L^1(E)$, e.g. $J = 0$, the following limit exists:

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to+\infty} \int_{|x|=r} \pi_{ij}\nu_e^j \, d\sigma_e.$$

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If we let $E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to+\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e$, then E and P fit together to given the *ADM energy-momentum four-vector* of the asymptotically flat end.

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What do we expect?

For reasonable matter, say corresponding to T satisfying an energy condition, we expect $E \geq 0$, and in $E \geq |P|$, i.e. the energy-momentum vector is future-pointing causal.

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This is the content of the *Positive Mass Theorem*.

Positive Mass Theorem

Recall that the dominant energy condition (DEC) implies on the initial data that $\rho \geq |J|$, i.e.

$$\frac{1}{2}(R(g) - \|K\|_g^2 + H^2) \geq |\mathrm{div}_g \pi|.$$

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Notes

- Riemannian case: Schoen-Yau, 1979 ($3 \leq n \leq 7$), Witten 1981 (spin manifolds).
- PET: Schoen-Yau, 1981 ($n = 3$), Witten (spin); Eichmair ($4 \leq n \leq 7$).
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The rigidity statement in the PMT is: $E = |P|$ only in case M^n is a space-like hypersurface in Minkowski space-time $\mathbb{M}^{n+1}$ with induced metric and second fundamental form g and K.

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The proof we will sketch blends ideas from the original Schoen-Yau proof, with observations of Bray and Lohkamp on the asymptotics.

Riemannian case

Outline

- Use conformal deformation to reduce to vacuum case $R(g) = 0$.
- Normalize asymptotics I: harmonically flat near infinity
- Simplify: compactify all but one end
- Normalize asymptotics II: Schwarzschild (mass m) near infinity, keeping $R \geq 0$. (Bray)
- If $m < 0$, make the metric EUCLIDEAN near infinity, keeping $R \geq 0$. (Lohkamp)
- Compactify to make a compact manifold with non-flat metric with $R \geq 0$, which is a connected sum with torus. (Lohkamp)
- Argue this metric can be deformed to PSC. Then argue that this is a contradiction—topological obstruction to PSC.
- Rigidity case requires separate argument

Mass under a conformal change

Illustrate in case $n = 3$. Suppose g AF, $q > 1/2$.
$\tilde{g} = u^4 g_{ij}$, where $u = 1 + \frac{A}{|x|} + \cdots$ near infinity (implies drop off in derivatives too).

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The mass integrands are thus related by

$$\sum_{i,j=1}^{3}(\tilde{g}_{ij,i} - \tilde{g}_{ii,j}) = \sum_{i,j=1}^{3}(g_{ij,i} - g_{ii,j}) + 8A\sum_{j=1}^{3}\frac{x^j}{|x|^3} + O(|x|^{-q-2}).$$

The surface integral of the A-term is

$\frac{1}{16\pi}\int\limits_{|x|=r} 8A\sum_{j=1}^{3}\frac{x^j}{|x|^3}\frac{x^j}{|x|}\, d\sigma_e = \frac{1}{16\pi}\frac{8A}{r^2}\cdot 4\pi r^2 = 2A.$

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As a corollary, this shows the mass of the Schwarzschild metric is indeed $m(g_S) = m(g_E) + 2\frac{m}{2} = 0 + m = m.$

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Note that if $0 < u < 1$, then $A \leq 0$, so that $m(\tilde{g}) \leq m(g)$.

So if the PET works for $\tilde{g}$ (which we might have arranged to be simpler than g in some way), then the PET holds for g.

Laplacian in weighted spaces on AF manifolds

We will use conformal changes of metric to deform the scalar curvature, and so we will want to understand the behavior of the Laplace operator Δ_g, as for $u > 0$:

$$R(u^{4/(n-2)}g) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} (\Delta_g u - \frac{n-2}{4(n-1)} R(g) u).$$

In particular we will be deforming to zero scalar curvature, so we want to solve $0 = (\Delta_g - \frac{n-2}{4(n-1)} R(g))u$.

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And we want the solution to be positive, $u > 0$, u tends to 1 at infinity (in the relevant end), and if possible, we'd like for u to have at least a partial expansion $u(x) = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{-(n-1)})$ (along with derivative decay).

$n = 3$: $u(x) = 1 + \frac{A}{|x|} + O(|x|^{-2})$

Weighted spaces

(M, g) asymptotically flat. Take $\sigma \geq 1$ a smooth function which in AF coordinates in any end is $\sigma(x) = |x|$.
Define a weighted L^p norm, $p \geq 1$:

$$\|u\|_{L^p_{-\tau}}^p = \int_M (|u|\sigma^\tau)^p \sigma^{-n} \, dv_g.$$

Weighted Sobolev norm

$$\|u\|_{W^{k,p}_{-\tau}} = \sum_{|\gamma| \leq k} \|D^\gamma u\|_{L^p_{-|\gamma|-\tau}}.$$

This gives a Banach space $W^{k,p}_{-\tau}(M, g)$.

(Reference: Bartnik, CPAM, 1986, e.g.)

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Let $\sigma(x,y) = \min(\sigma(x), \sigma(y))$, and let $d(x,y)$ be geodesic distance.

$[f]_{\alpha,-\beta} = \sup\limits_{x \neq y} (\sigma(x,y))^{\alpha+\beta} \frac{|f(x)-f(y)|}{(d(x,y)^{\alpha}}$.

Weighted Hölder norm

$$\|f\|_{C^{k,\alpha}_{-\beta}} = \sum_{|\gamma| \leq k} \sup_{x \in M} \left((\sigma(x))^{\beta+|\gamma|} |D^{\gamma} f(x)| \right) + [D^k f]_{\alpha,-\beta-k}.$$

Weighted spaces

A version of the Sobolev embedding is given in the following estimate:

Sobolev Embedding

If $1 \geq k - \frac{n}{p} \geq \alpha > 0$, then $\|u\|_{C^{0,\alpha}_{-\tau}} \leq C\|u\|_{W^{k,p}_{-\tau}}$. If $k > \frac{n}{p}$, $u \in W^{k,p}_{-\delta}$, then $|u(x)| = o(|x|^{-\delta})$.

Both sets of weighted spaces encode decay rates at infinity.

Behavior of Laplacian on weighted spaces

(M, g) AF, with suitable number of derivatives decaying suitably, $k \geq 2$, some range of $p > 1$ (and depends on decay rate of g).

Fredholm property

$\Delta_g : W^{k,p}_{-\delta} \to W^{k-2,p}_{-\delta-2}$ is Fredholm, provided $(-\delta) \notin \{m \in \mathbb{Z} : m \leq 2 - n, \ m \geq 0\}$.

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If $\delta \in (0, n-2)$, then Δ_g is invertible. (cf. No Euclidean harmonic functions decay at infinity weaker than $|x|^{2-n}$.) In this case we also have an estimate:

$$\|w\|_{W^{k,p}_{-\delta}} \leq C \|\Delta_g w\|_{W^{k-2,p}_{-\delta-2}}$$

Laplacian on weighted spaces

Conformal Laplacian

For g AF, the map $u \in X^k \to R(g)u \in Y^{k-2}$ is compact in weighted spaces: domain has k derivatives, say, while range has $(k-2)$ derivatives, say.

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Why?

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Why? Ascoli-Arzelà
So $\Delta_g - c(n)R(g)$ will be Fredholm of the same index as Δ_g on the suitable weighted spaces.

Expansion

Let $\delta \in (0, n-2)$. If (M, g) AF, and suppose an equation with Δ_g can be re-written as $\Delta v = F \in W^{2,q}_{-\delta^*-2}$, (for suitable $\tau > 0$, $\delta^* = \delta + \tau$) so that $\Delta : W^{2,q}_{-\delta^*} \to W^{2,q}_{-\delta^*-2}$ is Fredholm, there is a $w \in W^{2,q}_{-\delta^*}$ so that $\Delta(v - w) = 0$ for $|x| > R_0$, say. (Why?)

Then $(v - w)$ goes to zero and is harmonic at infinity, which means $(v - w) = \frac{A}{|x|^{n-2}} + O(|x|^{-(n-1)})$.

Improve estimate on w to get right error term in expansion.

(Cf. Bartnik—weighted Sobolev spaces; N. Meyers, 1963, weighted Hölder spaces)