

SUMMER 2012 MSRI SGW ON MATHEMATICAL GENERAL RELATIVITY  
SOME SCALAR CURVATURE BASICS

Recall that for better or for worse, I defined the curvature tensor following DoCarmo or O’Neill as  $R(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$ , with  $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = R_{ijk}^\ell \frac{\partial}{\partial x^\ell}$ .

Also, recall that a comma in a subscript denotes partial differentiation (with respect to some coordinate), whereas a semicolon in a subscript denotes covariant differentiation. For example, if  $h$  is a  $(1, 2)$ -tensor with components in a coordinate chart  $h_{jk}^i$ , then the covariant derivative  $\nabla h$  is a  $(1, 3)$ -tensor with components

$$h_{jk;\ell}^i := \nabla h\left(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) = \left(\nabla_{\frac{\partial}{\partial x^\ell}} h\right)\left(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = h_{jk,\ell}^i + \Gamma_{m\ell}^i h_{jk}^m - \Gamma_{j\ell}^m h_{mk}^i - \Gamma_{k\ell}^m h_{jm}^i.$$

**PROBLEM 1. LINEARIZATION OF SCALAR CURVATURE.** Let  $R(g) = g^{ij} R_{ij}$  be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation  $g(t) = g + th$  of  $g$  in the direction of a symmetric  $(0, 2)$ -tensor  $h$  (more generally, note that all you will use is that  $g(t)$  is smooth in  $t$ , with  $g(0) = g$  and  $g'(0) = h$ ). For small  $t$ ,  $g(t)$  is a metric. Define  $L_g(h) := DR_g(h) = \frac{d}{dt}\Big|_{t=0} R(g(t))$ .

Derive the scalar curvature formula

$$R(g) = g^{ij} R_{ij} = g^{ij} \left( \Gamma_{ij,k}^k - \Gamma_{ik,j}^k + \Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{j\ell}^k \Gamma_{ik}^\ell \right)$$

and use it to verify the identity

$$L_g(h) = -\Delta_g(\text{tr}_g(h)) + \text{div}_g(\text{div}_g(h)) - \langle h, \text{Ric}(g) \rangle_g.$$

**HINTS:** Compute in normal coordinates at a point  $p$ , so that  $g_{ij}(p) = \delta_{ij}$ , as well as  $\Gamma_{ij}^k(p) = 0$ , equivalently,  $g_{ij,k}(p) = 0$ . Indicated below are some formulas to guide you, and which *you should verify* as you compute. We emphasize here and below that all quantities are evaluated at  $p$ . In such normal coordinates, we have at  $p$ :  $h_{ij;k\ell} = h_{ij,k\ell} - h_{mj} \Gamma_{ik,\ell}^m - h_{im} \Gamma_{jk,\ell}^m$ . From this, one shows that at  $p$ ,

$$\begin{aligned} -\Delta_g(\text{tr}_g(h)) &= -g^{k\ell} g^{ij} h_{ij;k\ell} = -\sum_{i,k} (h_{ii,kk} - 2h_{km} \Gamma_{ik,i}^m) \\ \text{div}_g(\text{div}_g(h)) &= g^{j\ell} g^{ik} h_{ij;k\ell} = \sum_{i,k} (h_{ik,ik} - h_{km} \Gamma_{ii,k}^m - h_{km} \Gamma_{ik,i}^m). \end{aligned}$$

To find the variation of the scalar curvature, express the scalar curvature in terms of Christoffel symbols, and take a time derivative, expand and turn the crank. To find the derivative of the inverse metric, note that if  $A(t)$  is a smooth curve in  $GL(n)$ , then  $\frac{d}{dt} A^{-1}(t)$  is easily computed from  $A(t)A^{-1}(t) = I$  using the product rule.

Another approach is to note that  $\frac{d}{dt}\Big|_{t=0} \Gamma_{ij}^k$  are the components  $(\delta\Gamma)_{ij}^k$  of a tensor  $\delta\Gamma$  (this follows from #4 a on the earlier set on basic geometry problems), so that clearly the variation of the Ricci tensor is given by  $\frac{d}{dt}\Big|_{t=0} R_{ij} = (\delta\Gamma)_{ij;k}^k - (\delta\Gamma)_{ik;j}^k$ . One should now express  $\delta\Gamma$  in terms of the covariant derivative of  $h$ .

PROBLEM 2. CONFORMAL DEFORMATION OF SCALAR CURVATURE.

a. Suppose  $(M^n, g)$  is a Riemannian metric and  $\tilde{g} = e^\varphi g$ . Show that

$$R(\tilde{g}) = e^{-\varphi} \left( R(g) - (n-1)\Delta_g \varphi - \frac{1}{4}(n-1)(n-2)|\nabla \varphi|_g^2 \right).$$

b. In case  $n \geq 3$ , if we write  $e^\varphi = u^{\frac{4}{n-2}}$  for  $u > 0$ , then

$$R(\tilde{g}) = u^{-\frac{n+2}{n-2}} \left( R(g)u - \frac{4(n-1)}{(n-2)}\Delta_g u \right).$$

c. Let  $c(n) = \frac{n-2}{4(n-1)}$  and  $L_g u = \Delta_g u - c(n)R(g)u$  is the *conformal Laplacian*, show that the total scalar curvature of  $\tilde{g} = u^{\frac{4}{n-2}}g$  is given by

$$\int_M R(\tilde{g}) dv_{\tilde{g}} = c(n)^{-1} \int_M (|\nabla u|_g^2 + c(n)R(g)u^2) dv_g.$$

HINT: Show that  $dv_{\tilde{g}} = u^{\frac{2n}{n-2}} dv_g$ .

PROBLEM 3. VOLUME EXPANSION OF GEODESIC BALLS.

a. Suppose  $(V, \langle \cdot, \cdot \rangle)$  is an  $n$ -dimensional real inner product space. Suppose that  $T : V \rightarrow V$  is a self-adjoint linear operator. If  $d\sigma$  is the Euclidean area measure,  $\mathbb{B}^n$  is the unit ball, and  $\mathbb{S}^{n-1} \subset V$  is the unit sphere in  $V$ , then if  $\text{vol}$  is the Euclidean volume,

$$\int_{x \in \mathbb{S}^{n-1}} \langle T(x), x \rangle d\sigma = \text{trace}(T) \text{vol}(\mathbb{B}^n).$$

b. If  $(M, g)$  is Riemannian and  $p \in M$ , let  $B_r(p) \subset M$  be the geodesic ball of radius  $r > 0$  (for sufficiently small  $r$ ). Then

$$\text{vol}_g(B_r(p)) = \text{vol}(\mathbb{B}^n)r^n \left[ 1 - \frac{R(g)|_p}{6(n+2)} r^2 + O(r^3) \right].$$

HINT: You may wish to observe and use  $\det(I + tA) = 1 + t \text{trace}(A) + O(t^2)$ , along with Problem # 4c.

PROBLEM 4. METRIC EXPANSION IN NORMAL COORDINATES. Suppose that  $\gamma(t)$  is a unit-speed geodesic, and that  $J(t)$  is a Jacobi field along  $\gamma$ :  $J''(t) + R(\gamma'(t), J(t), \gamma'(t)) = 0$ .

a. If  $R$  is the Riemann curvature tensor, show that

$$J'''(t) + (\nabla_{\gamma'(t)} R)(\gamma'(t), J(t), \gamma'(t)) + R(\gamma'(t), J'(t), \gamma'(t)) = 0.$$

b. Suppose  $J(0) = 0$ . Let  $\chi(t) = \langle J(t), J(t) \rangle$ . Derive the fourth-order Taylor expansion

$$\chi(t) = \sum_{k=0}^4 \frac{\chi^{(k)}(0)}{k!} t^k + \mathcal{E}(t) = |J'(0)|^2 t^2 - \frac{1}{3} \langle R(\gamma'(0), J'(0), \gamma'(0)), J'(0) \rangle t^4 + O(t^5).$$

c. Suppose  $(M, g)$  is a Riemannian manifold and  $p \in M$ . Show that in normal coordinates centered at  $p$  (so  $x^i(p) = 0$ )

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikj\ell}x^kx^\ell + O(|x|^3).$$

HINT: See Lee Problem 10-3, and Lemma 10.7, but recall the curvature convention, and part b. should be useful.

PROBLEM 5. GEOMETRIC FORMULA FOR GAUSSIAN CURVATURE. Let  $(M, g)$  be a surface with a Riemannian metric  $g$ . Consider an orthonormal basis  $\mathbf{e}_1, \mathbf{e}_2$  of  $T_pM$ . Note that the Gauss curvature at  $p$  is just  $K(p) = R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2)$ , where  $R$  is the Riemann tensor. Consider a normal neighborhood of radius  $a > 0$  about  $p$ , with normal coordinates  $(x, y)$  built off of the orthonormal basis  $\mathbf{e}_1, \mathbf{e}_2$  of  $T_pM$ :  $(x, y) \mapsto \exp_p(x\mathbf{e}_1 + y\mathbf{e}_2)$ . Define geodesic polar coordinates by  $(r, \theta) \mapsto f(r, \theta) = \exp_p(r \cos \theta \mathbf{e}_1 + r \sin \theta \mathbf{e}_2)$ . Note that the change of coordinates map is just  $(r, \theta) \mapsto (x, y) = (r \cos \theta, r \sin \theta)$ , which shows that the map  $f$ , which is clearly smooth for  $r < a$  and all  $\theta$ , is a diffeomorphism for  $0 < r < a$  and  $\theta \in I$ , where  $I$  is any open interval of length at most  $2\pi$ . Note that by the Gauss' Lemma, the metric components in geodesic polar coordinates are  $g_{rr} = 1$ ,  $g_{r\theta} = 0$  and  $g_{\theta\theta} = \left| \frac{\partial f}{\partial \theta} \right|^2$ . Since the radial curves of constant  $\theta$  on  $M$  are geodesics, for any  $\theta_0$ ,  $J(r) = \frac{\partial f}{\partial \theta}(r, \theta_0)$  is a Jacobi field along the radial geodesic  $r \mapsto \gamma(r) = f(r, \theta_0)$ .

a. For any  $\theta_0$ , show that  $J'(0) \perp \gamma'(0)$ .

b. Use Problem 4 to show that  $g_{\theta\theta}(r, \theta) = r^2 - \frac{K(p)}{3}r^4 + \mathcal{E}(r, \theta)$ , where  $\mathcal{E}(r, \theta) = O(r^5)$  uniformly in  $\theta$ , i.e.  $R(r, \theta) \leq Cr^5$ ,  $C$  can be chosen independent of  $r$  and  $\theta$ . Use the Taylor expansion  $(1+x)^\alpha = 1 + \alpha x + O(x^2)$  to derive

$$\sqrt{g_{\theta\theta}}(r, \theta) = r - \frac{K(p)}{3!}r^3 + O(r^4).$$

c. Let  $L(r)$  be the length of a geodesic circle of radius  $r$  about  $p$ , and let  $A(r)$  be the area enclosed by this circle, both computed using the metric  $g$ . Show that

$$\lim_{r \rightarrow 0^+} \frac{3}{\pi} \frac{2\pi r - L(r)}{r^3} = K(p) = \lim_{r \rightarrow 0^+} \frac{12}{\pi} \frac{\pi r^2 - A(r)}{r^4}.$$

d. Let  $D$  be the unit disk in the plane:  $D = \{(x, y) : x^2 + y^2 < 1\}$ . Consider the hyperbolic metric

$$g_H = \frac{4}{(1 - (x^2 + y^2))^2} (dx^2 + dy^2) = \frac{4}{(1 - \rho^2)^2} (d\rho^2 + \rho^2 d\theta^2)$$

where  $(\rho, \theta)$  are standard polar coordinates on  $D$ . By solving the differential equation  $\frac{2d\rho}{1-\rho^2} = dr$ , show how to re-write the hyperbolic metric as  $g_H = dr^2 + \sinh^2 r d\theta^2$ . Use this along with the formulas above to show  $K = -1$  at the origin of coordinates (of course,  $K = -1$  everywhere). (Recall that  $dr^2 = dr \otimes dr$ .)