Problem: Prove that $B_T(x_c) \to 0$.

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Consider a self-avoiding walk (SAW) in two dimensions, in a strip of width T. We define a *spanning bridge* (so called as the bridge must fully span the strip) as a SAW whose origin has unique minimal x co-ordinate (w.l.o.g. $x = 0$), and whose end point has (not necessarily unique) maximal x co-ordinate, necessarily $x = T$. Define

$$B_T(x) = \sum_{n \geq T} b_n^{(T)} x^n,$$

where $b_n^{(T)}$ is the number of spanning bridges of n steps in a strip of width T. Note that $b_n^{(T)} = 0$, for $n < T$. It is well known that

$$B_T(x) \sim \frac{c(T)}{1 - x/x_c(T)},$$

where $x_c(T) > x_c(T+1) > \cdots > x_c$. Here x_c is the bulk critical point. For example, for the honeycomb lattice $x_c = 1/\sqrt{2+\sqrt{2}}$ as recently proved by Duminil-Copin and Smirnov [1]. For the honeycomb lattice, it is readily shown [1] that

$$B_T(x) \leq \left(\frac{x}{x_c}\right)^T B_T(x_c) \leq \left(\frac{x}{x_c}\right)^T.$$

Clearly, for $x < x_c$, $\lim_{T\to\infty} B_T(x) = 0$.

It is also expected that $\lim_{T\to\infty} B_T(x_c) = 0$, but this has not been proved, and seems difficult to prove. There is overwhelming numerical evidence, as well as persuasive SLE arguments that suggest that this is true. It is expected from SLE [1] that $B_T(x_c) \sim \frac{const.}{T^{1/4}}$, but this has not been proved. It is easy to prove that if the limit is zero, the limit is approached no faster than $1/T$.

Challenge: Prove that $\lim_{T\to\infty} B_T(x_c) = 0$.

References

[1] H Duminil-Copin and S Smirnov, *The connective constant of the honeycomb lattice is* $\sqrt{1+\sqrt{2}}$. arXiv 1007.0575