Causality in a Spacetime I

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Mathematical General Relativity

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M: an arbitrary spacetime (without the Einstein Equation imposed)

Let $p, q \in M$ (a priori p and q can be the same):

- $p \ll q$ means there exists a *future-directed timelike curve* from p to q.
- $p < q$ means there exists a *future-directed causal curve* from p to q.

All causal curves in our discussion are piecewise smooth curves.

We also use the notation $p \leq q$ to mean $p < q$ or $p = q$.

A basic result:

"*Suppose* γ *is a future directed causal curve from* p *to* q. *If* γ *is not a smooth null geodesic, then* γ *can be deformed to a nearby future directed timelike curve from* p *to* q."

Therefore, $p \ll q, q < r \Rightarrow p \ll r$.
Similarly, $p < q, q \ll r \Rightarrow p \ll r$.

The above lemma can be strengthen to

Proposition

A future directed causal curve from p *to* q *can be deformed to a nearby future directed timelike curve via fixed point variation unless* γ *is a smooth null geodesic along which there is no conjugate point of* p *before* q.

The **timelike future** of p is defined as $I^+(p) = \{q \in M \mid p \ll q\}$.

The **causal future** of p is defined as $J^+(p) = \{q \in M \mid p \leq q\}$.

If $A \subset M$, the **timelike future** and the **causal future** of A is

$$I^+(A) = \cup_{p \in A} I^+(p) \text{ and } J^+(p) = \cup_{p \in A} J^+(p).$$

Clearly, $(A \cup I^+(A)) \subset J^+(A)$. One defines the timelike past and the causal past $I^-(p), J^-(p), I^-(A), J^-(A)$ in a time-dual manner.

As a corollary of the previous proposition, one knows

" $q \in J^+(p) \setminus (I^+(p) \cup \{p\}) \Rightarrow$ *every future causal curve from p to q is a smooth future directed null geodesic along which p has no conjugate point before q.*"

Let U be an open set containing p.

$I^+(p, U) :=$ the set of points q in U such that there exists a future directed timelike curve $\gamma \subset U$ that goes from p to q. Similarly, one defines $J^+(p, U)$.

Lemma

Let U be an convex open set containing p. Then

- *$q \in I^+(p, U) \Leftrightarrow$ the unique geodesic $\sigma_{pq} \subset U$ from p to q is future timelike. Consequently, $I^+(p, U)$ is open in U.*
- *$q \in J^+(p, U) \setminus \{p\} \Leftrightarrow$ the unique geodesic $\sigma_{pq} \subset U$ from p to q is future null. Hence, $J^+(p, U)$ is the closure in U of $I^+(p, U)$.*

Proposition

For any $A \subset M$,

- $I^+(A)$ *is always an open set.*
- $p \in \partial I^+(A) \Rightarrow I^+(p) \subset I^+(A)$ and $I^-(p) \subset M \setminus \overline{I^+(A)}$.
- $I^+(A) = \mathrm{int}(J^+(A))$ and $J^+(A) \subset \overline{I^+(A)}$.
- *if $J^+(A)$ is a closed set, then*

$$J^+(A) \setminus I^+(A) = \partial J^+(A) \quad \text{and} \quad \partial J^+(A) = \partial I^+(A).$$