Second Variation of Area

Consider an isometric immersion $f : \Sigma^k \to M^n$ $(k < n)$. We write the metric on M using brackets $\langle \cdot, \cdot \rangle$. A smooth *variation* of f is a smooth map $F : I \times \Sigma \to M$ so that each $f_t := F(t, \cdot)$ is an immersion, $f_0 = f$; here I is an interval containing 0. We consider Σ compact, or if Σ is not compact, we consider only compactly supported variations (where F is constantly f outside a compact subset of Σ). Under such circumstances, we note that for any smooth map $F : I \times \Sigma \to M$ which agrees with f at time 0, we can restrict to a smaller interval $J \ni 0$ on which each f_t will be an immersion.

We define the variation field $V = F_*\big(\frac{\partial}{\partial t}|_{t=0}\big)$, the pushforward of the time direction; V is a section of the pullback bundle $f^*(TM)$. We let F_i denote the pushforward $F_*\big(\frac{\partial}{\partial x^i}\big)$, and $F_t = F_*(\frac{\partial}{\partial t})$. Let $g(t)$ be the metric induced on Σ from the immersion f_t, and let dA_t denote the induced volume measure, $dA_\Sigma := dA_0$. In coordinates we have

$$g_{ij}(t) = \langle F_i, F_j \rangle.$$

We use the notation $\overline{\nabla}_i Y := \overline{\nabla}_{F_i} Y$, and $\overline{\nabla}_t Y := \overline{\nabla}_{F_t} Y$ for vector fields Y on M, and this is well-defined by the pointwise nature of $\overline{\nabla}_X Y$ with respect to X.

Technical Remark. The method of computation to follow is valid, strictly speaking, at points (t_0, p) where F is an immersion on $I \times \Sigma$. Assuming this, given any vector field X defined along the map F, i.e. $X : I \times \Sigma \to TM$ with $X(t_0, p) \in T_{F(t_0,p)}M$, we can extend the field to a field X on M, agreeing with the original near the point $F(t_0, p)$. We can then use the covariant derivative identity

$$\frac{\overline{D}X}{dt} = \overline{\nabla}_t X.$$

Near such $F(t_0, p)$, consider extensions of F_i and F_j, and note that $g_{ij}(t, q) = \langle F_i, F_j \rangle \circ F(t, q)$ near (t_0, p). Thus

$$\dot{g}_{ij} = F_t\big(\langle F_i, F_j \rangle\big) = \langle \overline{\nabla}_t F_i, F_j \rangle + \langle F_i, \overline{\nabla}_t F_j \rangle.$$

The formulas below will also hold at points where F fails to be an immersion. There are several ways to think about this, whether by a continuity argument, or approximating F by an immersion locally, or by considering (as in Spivak Vol. IV) the map $\Phi : I \times \Sigma \to M \times \mathbb{R}$ given by $\Phi(t, p) = (F(t, p), t)$, and extending the metric on M to the product metric, still written "$\langle\ ,\ \rangle$." Φ is an immersion, so the computations will hold using Φ. It is not hard to check from the product structure that the variation formulas are the ones proved below.

The area functional we want to vary is $A(t) = \int\limits_\Sigma dA_t$. We note that the volume element in local coordinates is just $dA_t = \sqrt{\det g(t)}\, dx$. By a simple partition-of-unity argument, we see that the variation is given by

$$A'(t) = \int\limits_\Sigma \frac{d}{dt} \sqrt{\det g(t)}\, dx$$

i.e. the value of the variation is an integral whose integrand has the above local expression, which can thus be given a global expression with some work, which we now carry out.

From Cramer's rule and the symmetry of the metric g, we have (in any coordinates)

$$\frac{\partial \det g}{\partial g_{ij}} = (\det g) g^{ij}.$$

From this it is easy to see

$$\frac{d}{dt}\sqrt{\det g(t)} = \frac{1}{2}\sqrt{\det g(t)}\; g^{ij}\dot{g}_{ij}.$$

As discussed above, we have $\dot{g}_{ij} = \langle \overline{\nabla}_t F_i, F_j\rangle + \langle F_i, \overline{\nabla}_t F_j\rangle$. We now combine the fact that the Levi-Civita connection is torsion-free along with the identity $0 = F_*[\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}] = [F_t, F_i]$, and use the symmetry of the metric to obtain (recall we use the Einstein summation convention)

$$\frac{d}{dt}\sqrt{\det g(t)} = \sqrt{\det g(t)}\; g^{ij}\langle \overline{\nabla}_t F_i, F_j\rangle = \sqrt{\det g(t)}\; g^{ij}\langle \overline{\nabla}_i F_t, F_j\rangle = \mathrm{div}_\Sigma(F_t)\sqrt{\det g(t)}.$$

If we evaluate this at $t = 0$, we obtain

$$\frac{d}{dt}\Big|_{t=0}\sqrt{\det g} = \mathrm{div}_\Sigma(V)\sqrt{\det g} = \big(\mathrm{div}_\Sigma(V^T) + \mathrm{div}_\Sigma(V^N)\big)\sqrt{\det g(t)}$$

where V^T and V^N are the tangential and normal components of V respectively. We note that we can apply the Divergence Theorem to the $\mathrm{div}_\Sigma(V^T)$-term. For the other term we note

$$\mathrm{div}_\Sigma(V^N) = \sum_{i=1}^{k}\langle \overline{\nabla}_{E_i} V^N, E_i\rangle = -\langle V, \sum_{i=1}^{k}\big(\overline{\nabla}_{E_i} E_i\big)^N\rangle = -\langle V, \mathbf{H}\rangle.$$

Putting this all together, we have arrived at the **First Variation Formula**:

$$A'(0) = -\int_\Sigma \langle V, \mathbf{H}\rangle\; dA_\Sigma + \int_{\partial\Sigma}\langle V, \eta\rangle\; d\sigma.$$

η is the outer unit co-normal along $\partial\Sigma$, and the integrals are done with respect to the measures induced by the metric on M.

Next, it's onto the second variation; we compute this at a *minimal* immersion of Σ. We first note the formula for the derivative of the inverse of a matrix yields the identity $\frac{d}{dt}(g^{ij}) := \dot{g}^{ij} = -g^{ik}\dot{g}_{k\ell}g^{\ell j}$. Using this we make the following computation:

$$\begin{aligned}\frac{d^2}{dt^2}\sqrt{\det g(t)} &= \frac{d}{dt}\Big(\frac{1}{2}\sqrt{\det g(t)}\; g^{ij}\dot{g}_{ij}\Big)\\ &= \frac{1}{4}\sqrt{\det g(t)}\,\big(g^{ij}\dot{g}_{ij}\big)^2 - \frac{1}{2}\sqrt{\det g(t)}\; g^{ik}\dot{g}_{k\ell}g^{\ell j}\dot{g}_{ij} + \frac{1}{2}\sqrt{\det g(t)}\; g^{ij}\ddot{g}_{ij}.\end{aligned}$$

We examine each of these three terms in turn, at $t = 0$, and we use identities derived above. First we have

$$\frac{1}{4}\sqrt{\det g}\,\big(g^{ij}\dot{g}_{ij}\big)^2 = \frac{1}{4}\big(2\mathrm{div}_\Sigma V\big)^2\sqrt{\det g} = \big(\mathrm{div}_\Sigma V\big)^2\sqrt{\det g}.$$

If the immersion is minimal, then note that this last term is just $\big(\mathrm{div}_\Sigma V^T\big)^2\sqrt{\det g}$, since $\mathrm{div}_\Sigma V^N = -\langle V, \mathbf{H}\rangle = 0$ by minimality.

As for the second term, we have

$$\begin{aligned}g^{ik}\dot{g}_{k\ell}g^{\ell j}\dot{g}_{ij} &= g^{ik}\big(\langle \overline{\nabla}_k V, F_\ell\rangle + \langle F_k, \overline{\nabla}_\ell V\rangle\big)g^{\ell j}\big(\langle \overline{\nabla}_i V, F_j\rangle + \langle F_i, \overline{\nabla}_j V\rangle\big)\\ &= \sum_{i,j=1}^{k}\big(\langle \overline{\nabla}_{E_i} V, E_j\rangle + \langle E_i, \overline{\nabla}_{E_j} V\rangle\big)^2\end{aligned}$$

where we have used orthonormal frame fields E_i to simplify things; this is valid since we are just contracting tensors ($\dot{g}$ is a tensor, and its expression in the first equation is indeed tensorial in the F_k, F_ℓ slots); we could also derive this last expression from scratch, identifying $E_i = F_*(e_i)$, where e_i is defined on $I \times \Sigma$ and is a frame field on Σ for each t, with $[e_i, \frac{\partial}{\partial t}] = 0$.

Finally we come to the last term, which is interesting since curvature will be involved. Indeed we have

$$\begin{aligned}\ddot{g}_{ij} &= \langle \overline{\nabla}_t \overline{\nabla}_t F_i, F_j\rangle + 2\langle \overline{\nabla}_t F_i, \overline{\nabla}_t F_j\rangle + \langle F_i \overline{\nabla}_t \overline{\nabla}_t F_j\rangle \\ &= \langle \overline{\nabla}_t \overline{\nabla}_i F_t, F_j\rangle + 2\langle \overline{\nabla}_i F_t, \overline{\nabla}_j F_t\rangle + \langle F_i, \overline{\nabla}_t \overline{\nabla}_j F_t\rangle.\end{aligned}$$

Notice that we used the fact that derivatives of a mapping commute: denoting F_t and F_i with partials, we have

$$\frac{\overline{D}}{dt}\frac{\partial F}{\partial x^i} = \frac{\overline{D}}{dx^i}\frac{\partial F}{\partial t}.$$

Of course when we commute the covariant derivatives on a vector field, though, we get the curvature $\overline{R}$ of M:

$$\overline{R}(X,Y)Z = \overline{\nabla}_{[X,Y]}Z - \left[\overline{\nabla}_X, \overline{\nabla}_Y\right] Z.$$

Thus we get at $t = 0$ (again using $[F_i, F_t] = 0$)

$$\ddot{g}_{ij} = \langle \overline{R}(F_i, V)V, F_j\rangle + \langle \overline{\nabla}_i \overline{\nabla}_t F_t, F_j\rangle + \langle \overline{\nabla}_j \overline{\nabla}_t F_t, F_i\rangle + 2\langle \overline{\nabla}_i V, \overline{\nabla}_j V\rangle + \langle \overline{R}(F_j, V)V, F_i\rangle.$$

From this it follows

$$\frac{1}{2} g^{ij}\ddot{g}_{ij} = \sum_{i=1}^{k} |\overline{\nabla}_{E_i} V|^2 - \sum_{i=1}^{k} \langle \overline{R}(V, E_i)V, E_i\rangle + \mathrm{div}_\Sigma(\overline{\nabla}_V F_t).$$

The last term is just $\mathrm{div}_\Sigma(\overline{\nabla}_V F_t)^T$, since

$$\mathrm{div}_\Sigma(\overline{\nabla}_V F_t)^N = -\langle \overline{\nabla}_V F_t, \mathbf{H}\rangle = 0$$

by minimality again. This last term then yields a term of the form $\int_{\partial\Sigma} \langle \overline{\nabla}_V F_t, \nu\rangle \, d\sigma$ in the Second Variation Formula, the rest of which we get by adding up the other contributions. Instead of jotting this down, we turn to getting a useful form of the formula in the case of *normal variations*.

When the variation field V is normal to Σ, we have several simplifications. First we have

$$\sum_{i,j=1}^{k} \left(\langle \overline{\nabla}_{E_i} V, E_j\rangle + \langle E_i, \overline{\nabla}_{E_j} V\rangle\right)^2 = \sum_{i,j=1}^{k} \left(-\langle V, \overline{\nabla}_{E_i} E_j\rangle - \langle \overline{\nabla}_{E_j} E_i, V\rangle\right)^2 = 4\sum_{i,j=1}^{k} \langle V, \overline{\nabla}_{E_i} E_j\rangle^2$$

and this last term is just $4|A^V|^2$, where A^V is the symmetric $(0,2)$-tensor which is the inner prouct of the second fundamental form $A(X,Y) = (\overline{\nabla}_X Y)^N$ with V: $A^V(X,Y) = \langle V, A(X,Y)\rangle$. Note that in the last equation above we used the fact that V is normal, and the second fundamental form is symmetric, namely $\overline{\nabla}_{E_i} E_j - \overline{\nabla}_{E_j} E_i = [E_i, E_j]$, which is orthogonal to V.

Denoting by ∇^N the normal connection on the normal bundle $N\Sigma$ (whose definition you will recall from the formula below), we have

$$\begin{aligned}\sum_{i=1}^{k} |\overline{\nabla}_{E_i} V|^2 &= \sum_{i=1}^{k} \left(|\nabla^N_{E_i} V|^2 + |(\overline{\nabla}_{E_i} V)^T|^2 \right) \\ &= \sum_{i=1}^{k} |\nabla^N_{E_i} V|^2 + \sum_{i,j=1}^{k} \langle \overline{\nabla}_{E_i} V, E_j \rangle^2 \\ &= |\nabla^N V|^2 + |A^V|^2.\end{aligned}$$

Finally we introduce the normal Ricci curvature form $\mathcal{R}$, an operator on the normal bundle on Σ, defined for vectors W normal to Σ by

$$\mathcal{R}(W) = \sum_{i=1}^{k} \left(\overline{R}(E_i, W) E_i \right)^N$$

which is well-defined, independent of the frame chosen. We thus arrive at the **Second Variation Formula** at a minimal submanifold for *normal variations*:

$$A''(0) = \int_{\Sigma} \left[|\nabla^N V|^2 - |A^V|^2 - \langle \mathcal{R}(V), V \rangle \right] dA_{\Sigma} + \int_{\partial\Sigma} \langle \overline{\nabla}_V F_t, \nu \rangle \, d\sigma.$$

In the two-sided hypersurface case, we write $V = \varphi\nu$, where ν is a unit normal field to Σ, $\mathbf{H} = H\nu$, and for simplicity we take $\varphi = 0$ on $\partial\Sigma$. Then $A'(0) = -\int_{\Sigma} \varphi H \, dA_{\Sigma}$. (Beware—some authors pick $\mathbf{H} = -H\nu$...) Moreover, for X tangent to Σ, $\nabla^N_X V = (X(\varphi)\, \nu + \varphi \overline{\nabla}_X \nu)^N = X(\varphi)\nu$, so $|\nabla^N V|^2 = |\nabla^{\Sigma} f|^2$, $|A^V|^2 = \varphi^2 |A|^2$, so that

$$A''(0) = \int_{\Sigma} \left[|\nabla^{\Sigma}\varphi|^2 - \varphi^2 |A|^2 - \varphi^2 \mathrm{Ric}(\nu,\nu) \right] dA_{\Sigma} = -\int_{\Sigma} \varphi \mathcal{L} \varphi \, dA_{\Sigma},$$

where $\mathcal{L}$ is the *Jacobi operator* $\mathcal{L}\varphi = \Delta_{\Sigma}\varphi + (|A|^2 + \mathrm{Ric}(\nu,\nu))\varphi$.

Exercise What can you say about the second variation formula in the previous paragraph if H is constant (not necessarily 0)?

INTRO TO THE CONFORMAL METHOD

METHOD OF SUPER- AND SUB-SOLUTIONS

We want to study the following semilinear elliptic equation: $Pu = G(x,u)$. Here P is a second-order scalar elliptic operator, so it can be written

$$Pu = \sum_{i,j} a_{ij} D_{ij} u + \sum_i b_i D_i u + cu$$

where (a_{ij}) is a symmetric, positive-definite matrix; we also remark that $G(x,u)$ does not depend on the derivatives of u. We may be interested in solving such an equation on a bounded domain $\Omega \subset \mathbb{R}^n$, or on a closed Riemannian manifold (M^n, g), in which case the above form for P will hold in local charts, or on a bounded domain $\Omega \subset M$. In the case of a bounded domain Ω, one may also want to impose the boundary condition $u = \psi$ on $\partial\Omega$.

Definition. Functions ϕ_+ and ϕ_- satisfying

$$P\phi_+ \leq G(x,\phi_+) \quad , \quad P\phi_- \geq G(x,\phi_-)$$

along with a boundedness condition

$$-K \leq \phi_- \leq \phi_+ \leq K$$

for some K, are called *super*- and *sub-solutions* respectively. In the case we consider the boundary value problem above, we also require $\phi_- \leq \psi$ and $\phi_+ \geq \psi$ on $\partial\Omega$.

One can show that under certain conditions the existence of super- and sub-solutions implies the existence of a solution ϕ to the PDE which satisfies the bound $\phi_- \leq \phi \leq \phi_+$. We will state two sets of sufficient conditions and do the proof for one of them.

Theorem. (1) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose there is a Λ so that $|G(x,s)| \leq \Lambda$ for $-K \leq s \leq K$, and there is a $\mu \in (0,1)$ so that for $|s_i| \leq K$

$$|G(x_1,s_1) - G(x_2,s_2)| \leq \Lambda \left(|x_1 - x_2|^\mu + |s_1 - s_2|\right).$$

Assume also that the coefficients of P are in $C^{0,\mu}(\overline{\Omega})$, the domain has a smooth boundary ($C^{2,\mu}$). Then if there are super- and sub-solutions $\phi_\pm \in C^{2,\mu}(\overline{\Omega})$ for the boundary value problem, there is a solution $\phi \in C^{2,\mu}(\overline{\Omega})$ with $\phi_- \leq \phi \leq \phi_+$.

(2) Suppose $G \in C^1$, the coefficients of P are smooth (enough), and that either the domain Ω is a "nice" bounded domain or that we work on a closed Riemannian manifold (M^n, g). Assume also the existence of super- and sub-solutions $\phi_\pm \in W^{2,p}$ with $2 > \frac{n}{p} + 1$ (*i.e.* $p > n$) so that by the Sobolev embedding $\phi_\pm \in C^{1,\alpha}$ for $\alpha \in (0, 1 - \frac{n}{p})$. Then there is a solution $\phi \in W^{2,p}$ of the PDE with $\phi_- \leq \phi \leq \phi_+$. ($\phi$ will be more regular if the coefficients of P and G are more regular.)

Proof: This proof is an expanded version of the proof from [I]. Let ρ be a constant, let $L = -P + \rho$, and let $F(x,s) = -G(x,s) + \rho s$. The PDE can then be written $Lu = F(x,u)$. By choosing ρ sufficiently large, we can arrange that $F(x,s)$ is increasing in s, for $|s| \leq K$ (hence for $s \in [\min \phi_-, \max \phi_+]$), and that $(\rho - c) > 0$. We note that the supersolution satisfies

$$L\phi_+ = -P\phi_+ + \rho\phi_+ \geq -G(x,\phi_+) + \rho\phi_+ = F(x,\phi_+)$$

and similarly $L\phi_- \le F(x,\phi_-)$.

We also have a Maximum Principle: if $Lu \ge 0$, then $u \ge 0$. Indeed, suppose that u has a negative (interior) minimum at some x_0. Then

$$\begin{aligned} Lu(x_0) &= -\sum_{i,j} a_{ij}(x_0)D_{ij}u(x_0) - \sum_i b_i(x_0)D_iu(x_0) + (\rho - c)u(x_0) \\ &= -\sum_{i,j} a_{ij}(x_0)D_{ij}u(x_0) + (\rho - c)u(x_0) \end{aligned}$$

which is negative, by the choice of ρ (*i.e.* $(\rho - c) > 0$), a contradiction. Here we use the fact that the product of the positive-definite matrix $A = (a_{ij})$ and the positive semi-definite matrix $B = (D_{ij}u(x_0))$ is positive semi-definite; for a proof, one may diagonalize A and then apply a criterion for semi-definiteness to AB from [St], for example.

Remark. For the boundary problem, we let $v = u - \psi$ and solve $Lv = F(x,u) - L\psi$, $v = 0$ on $\partial\Omega$. For v which vanish on the boundary, the above argument still works to show $v \ge 0$ when $Lv \ge 0$.

We now construct a sequence of functions that will converge to a solution of the PDE.

Step 1: We construct a sequence $\{\phi_j\} \subset W^{2,p}$ satisfying, for $n = 1, 2, \ldots,$

$$\begin{aligned} L\phi_1 &= F(x,\phi_+) \\ L\phi_{j+1} &= F(x,\phi_j) \end{aligned}$$

This sequence satisfies a monotonicity property $\phi_+ \ge \phi_1 \ge \phi_2 \ge \ldots \ge \phi_-$.

Remark. For the boundary problem, we let $\phi_j = v_j + \psi$ and solve the following PDE with $v_j = 0$ on $\partial\Omega$:

$$\begin{aligned} Lv_1 &= F(x,\phi_+) - L\psi \\ Lv_{j+1} &= F(x,\phi_j) - L\psi. \end{aligned}$$

By the Maximum Principle, there is only one solution of the homogeneous equation, so the Fredholm Alternative gives a unique sequence v_j. In the remainder of the proof we assume we are in the case of a closed manifold, although the only real modifications to the proof involve invoking boundary estimates for the PDE [G-T].

Proof of Step 1: Existence of the sequence follows from the Fredholm Alternative/Hodge Decomposition. Indeed we show that the kernel of L^* is trivial. This follows from the Maximum Principle proved above (applied to u and $-u$ when $L^*u = 0$). For operators we will often be concerned with of the form $L = -\Delta_g + (\rho - c) = L^*$, we see that $Lu = 0$ implies (by multiplying by u and integrating by parts)

$$\int_M |\nabla u|^2 d\mu_g = \int_M (c - \rho)u^2 d\mu_g$$

from which we see $u \equiv 0$.

We remark that for smoother coefficients (say C^1 or even Lipschitz) we produce a sequence in $W^{3,p}$ (cf. [G-T] Ch. 9). This follows since $W^{2,p} \hookrightarrow C^{1,\alpha}$ so that $F(x, \phi_\pm)$ and then $F(x, \phi_j)$ are in $C^1 \subset W^{1,p}$.

Monotonicity follows from the Maximum Principle derived above. Indeed, we compute

$$L(\phi_+ - \phi_1) = L\phi_+ - L\phi_1 = L\phi_+ - F(x, \phi_+) \geq 0.$$

Thus we conclude as above that $(\phi_+ - \phi_1) \geq 0$. We seek to iterate this argument. We see that $L(\phi_1 - \phi_2) = F(x, \phi_+) - F(x, \phi_1) \geq 0$, since $\phi_+ \geq \phi_1$ and we arranged that F is increasing in the second variable. Thus we get $\phi_1 \geq \phi_2$, and inductively $\phi_j \geq \phi_{j+1}$.

Finally we see that $L\phi_1 - L\phi_- = F(x, \phi_+) - L\phi_- \geq F(x, \phi_+) - F(x, \phi-) \geq 0$, and hence $\phi_1 \geq \phi_-$. Similarly, $L\phi_{j+1} - L\phi_- \geq F(x, \phi_j) - F(x, \phi_-)$, which inductively is nonnegative.

Step 2: C^0 convergence.

Proof: By the previous step we have that the sequence $\{\phi_j\}$ is sup-bounded. We show equicontinuity, so that we can then invoke Arzela-Ascoli. By the elliptic estimate for operators without kernel (such as our L), we have

$$\|\phi_j\|_{W^{2,p}} \leq C\|L\phi_j\|_{L^p} = C\|F(x, \phi_{j-1})\|_{L^p} \leq C'.$$

By the Sobolev embedding ($p > n$), $\|\phi_j\|_{C^1} \leq C''\|\phi_j\|_{W^{2,p}}$, we then see that the sequence is indeed equicontinuous; we remark that equicontinuity follows even in case $p > \frac{n}{2}$ by the embedding $W^{2,p} \hookrightarrow C^{0,\beta}$ for $0 < \beta < \min(p - \frac{n}{2}, 1)$. In any case, by Arzela-Ascoli a subsequence converges in C^0; by monotonicity, then, the whole sequence converges to a function ϕ.

Step 3: Regularity of ϕ.

Proof: Given $\epsilon > 0$, we have by the elliptic estimate

$$\|\phi_l - \phi_j\|_{W^{2,p}} \leq C\|L(\phi_l - \phi_j)\|_{L^p} = C\|F(x, \phi_{l-1}) - F(x, \phi_{j-1})\|_{L^p} < \epsilon$$

for l,j large enough, by uniform continuity of F and C^0-convergence of $\{\phi_j\}$. This estimate shows that the sequence is Cauchy in $W^{2,p}$, so that $\phi \in W^{2,p} \hookrightarrow C^{1,\alpha}$; in the next step we use this convergence to show that ϕ solves the desired PDE. As such, the elliptic regularity shows that in fact $\phi \in C^{2,\alpha}$. On the other hand, we can see this another way. As $\phi_j \in C^{2,\alpha}$ (using elliptic regularity and the fact that $G(x, \phi_{j-1}) \in C^1$), we can use the Schauder estimates to get

$$\|\phi_l - \phi_j\|_{C^{2,\alpha}} \leq C\|L(\phi_l - \phi_j)\|_{C^{0,\alpha}} = C\|F(x, \phi_{l-1}) - F(x, \phi_{j-1})\|_{C^{0,\alpha}}.$$

We indicate how to show that $\{F(x, \phi_j)\}$ is $C^{0,\alpha}$-Cauchy, which then gives that ϕ_j converges in $C^{2,\alpha}$, and so then ϕ is $C^{2,\alpha}$. To do this let us assume we are working in a coordinate chart. We then write

$$\begin{aligned}
|F(x, \phi_l(x)) - F(x, \phi_j(x)) &- F(y, \phi_l(y)) + F(y, \phi_j(y))| \\
&= |F(x, \phi_l(x)) - F(y, \phi_l(x)) + F(y, \phi_l(x)) - F(y, \phi_l(y)) \\
&\quad + F(y, \phi_j(y)) - F(x, \phi_j(y)) + F(x, \phi_j(y)) - F(x, \phi_j(x))| \\
&\leq C\big[|x - y| + |\phi_l(x) - \phi_l(y)| + |\phi_j(x) - \phi_j(y)|\big] \\
&\leq C'|x - y|
\end{aligned}$$

We have used the uniform bounds on F and ϕ_j in C^1. From this estimate, given ϵ, we can bound the Hölder quotient by ϵ for $|x-y|$ sufficiently small, less than $\delta(\epsilon) = \left(\frac{\epsilon}{C'}\right)^{1/(1-\alpha)}$. For $|x-y|$ bigger than this small amount, we will have to choose l and j sufficiently large to bound the Hölder quotient, which we can do by staring at the following:

$$|F(x,\phi_l(x)) - F(x,\phi_j(x)) - F(y,\phi_l(y)) + F(y,\phi_j(y))| \leq C\left[|\phi_l(x)-\phi_j(x)| + |\phi_l(y)-\phi_j(y)|\right]$$

which can be made less than $\epsilon\delta(\epsilon)^\alpha$ for l, j sufficiently large.

Remark. If the coefficients of P were smoother than just Hölder continuous, than we could get a $W^{3,p} \hookrightarrow C^{2,\alpha}$-estimate and convergence as above. We can bootstrap to C^∞ provided the coefficients are smooth and that G is smooth (and the boundary (if present) is smooth).

Step 4: ϕ satisfies the PDE.

Proof: By the Sobolev embedding $W^{2,p} \hookrightarrow C^{1,\alpha}$, and the regularity of F, the map $W^{2,p} \ni u \mapsto F(x,u) \in L^p$ is continuous. Thus the map $(L-F) : W^{2,p} \to L^p$ is continuous, and so $(L-F)\phi = \lim_{j\to\infty}(L-F)\phi_j$. Moreover,

$$\|L\phi_j - F(x,\phi_j)\|_{L^p} = \|F(x,\phi_{j-1}) - F(x,\phi_j)\|_{L^p} \xrightarrow{j\to\infty} 0. \quad \blacksquare$$

THE CONFORMAL METHOD FOR THE VACUUM EINSTEIN CONSTRAINT EQUATIONS

We will now discuss the initial-value constraints for the Einstein equation in space-time dimension four. Recall that the Einstein tensor G for a metric $\bar{g}$ is given by the equation $G = Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g}$, and the vacuum Einstein equation reduces to $Ric(\bar{g}) = 0$ on a four-manifold $\mathcal{S}$ with Lorentzian metric $\bar{g}$.

If you write this equation out in harmonic (wave) coordinates, you will see that the PDE system is hyperbolic in nature (see Wald [W], or Intro GR Lecture Notes). So this leads one to ask whether an initial-value formulation of the Einstein equations can be given. In fact this is so, with the initial data being a three-manifold Σ, along with a Riemannian metric g and a symmetric $(0,2)$ (or $(2,0)$) tensor K. The picture we have in mind is that Σ^3 is an embedded hypersurface in a spacetime $(\mathcal{S}^4, \bar{g})$, where the induced metric g is Riemannian (so Σ is called spacelike) and the induced second fundamental form is K. If this were so, g and K have to satisfy certain compatibility equations which come from the Gauss and Codazzi equations for immersions, coupled with the Einstein equation.

Proposition. The vacuum Einstein constraint equations are

$$\begin{aligned} R(g) - |K|^2 + H^2 &= 0 \\ div_g K - dH &= 0 \end{aligned} \tag{1}$$

Here the quantities are computed with respect to g, and H is the mean curvature $H = g^{ij}K_{ij}$.

We see that these equations are given solely in terms of g and K, and not the ambient metric. So we forget about any ambient space, and look at data (Σ^3, g, K) as above, satisfying the Einstein constraints. It turns out that these constraints are also sufficient to prove the existence of a (and a unique "maximal" such) spacetime $(\mathcal{S}^4, \bar{g})$ satisfying the Einstein equations in vacuum, and containing Σ, with first and second fundamental form g and K respectively. (See [W]; the results are due to Choquet-Bruhat, Geroch, and earlier work by Leray.) The picture is that Σ is a snapshot at some initial time, and the Einstein equations can evolve the data from there. Although the initial data cannot be arbitrary, the equations above are the only constraints.

One goal is then to understand better how to describe the space of solutions of the constraints. There are lots of open questions, and indeed we'll only look at the special, well-understood case when Σ^3 is closed, and the mean curvature is constant (closed, CMC case). We'll see that in this restrictive case that there is a nice way to describe the solutions in terms of certain free data. Instead of trying to give g and K in one step, we will essentially give certain parts of g and K and solve for the rest. Indeed the constraints are a system of four equations for twelve unknowns, so some of the data should be free and other parts then determined by the system. The technique we describe here is the conformal method, based on work of Choquet-Bruhat, Lichnerowicz, York (see *e.g.* [Yo]).

The first step is the TT (Transverse-Traceless) decomposition of symmetric two-tensors. Given a Riemannian metric g on Σ^3, we can decompose any symmetric (0,2)-tensor (or (2,0)) Ψ as $\Psi = \Psi_{TT} + \Psi_L + \Psi_{Tr}$, where Ψ_{TT} is trace-free and divergence-free (transverse), and Ψ_{Tr} is pure-trace: $\Psi_{Tr} = \frac{1}{3}\left(Tr_g \Psi\right) g$. We now specify an ansatz for Ψ_L, and then find an equation for it by requiring Ψ_{TT} to be transverse-traceless.

Our ansatz for Ψ_L (the longitudinal part) is

$$(\Psi_L)_{ab} = L_g W := W_{a;b} + W_{b;a} - \frac{2}{3} g_{ab} \left(div_g W\right).$$

Here W is a vector field (or the associated form $W^\flat$, $W_a = g_{ab} W^a$), the operator L_g is called the conformal Killing operator, and elements in the kernel of L_g are called conformal Killing fields; such vector fields (CKV's) are infinitesimal conformal isometries (like Killing fields whose flows are isometries). Clearly $L_g W$ is trace-free, so that $\Psi_{TT} := \Psi - \Psi_L - \Psi_{Tr}$ is trace-free for any choice of W. We get an equation for W by requiring the divergence to vanish too:

$$-div_g(L_g W) = -div_g \left(\Psi - \frac{1}{3}\left(Tr_g \Psi\right) g \right).$$

Note that $\left(\Psi - \frac{1}{3}\left(Tr_g \Psi\right) g\right)$ is precisely the trace-free part of Ψ.

Proposition. Given Ψ, we can solve for W, uniquely up to conformal Killing fields, so that Ψ_L is uniquely determined. Indeed, $\mathcal{P} := -div_g \circ L_g$ is elliptic and self-adjoint, and the right-hand side of the equation above is orthogonal to the kernel of $\mathcal{P}$, which is precisely the space of conformal killing fields (CKV's).

Proof: By integration-by-parts we have

$$\begin{aligned}(W, \mathcal{P}W)_{L^2} &= (\nabla W, L_g W)_{L^2} \\ &= \frac{1}{2}\int_\Sigma (W_{a;b} + W_{b;a})\,(L_g W)^{ab} d\mu_g \\ &= \frac{1}{2}\,(L_g W, L_g W)_{L^2}\end{aligned}$$

The last two equations follow from the facts that $L_g W$ is symmetric and traceless, respectively. As a corollary, we see that indeed the kernel of $\mathcal{P}$ is precisely the kernel of L_g, and moreover that $\mathcal{P}$ is self-adjoint.

Now we show the ellipticity of the operator $\mathcal{P}$ which takes vector fields to vector fields. We write

$$(\mathcal{P}W)_a = -(div_g \circ L_g W)_a = -\left(W_{a;bc} + W_{b;ac} - \frac{2}{3} g_{ab} W^d_{\ ;dc}\right) g^{bc}.$$

So the principal symbol, up to a constant factor, is given as follows: for each cotangent vector $\xi \neq 0$ at a point $p \in \Sigma$, we get a map on $T_p\Sigma$ which sends

$$V \mapsto V^c \xi_a \xi_c + |\xi|^2 V - \frac{2}{3} V^c \xi_c \xi_a = |\xi|^2 V + \frac{1}{3}\xi(V)\xi^\sharp.$$

If V is in the kernel of this map, we see that

$$V = -\frac{1}{3}\frac{\xi(V)}{|\xi|^2}\xi^\sharp$$

to which we apply ξ and get

$$\xi(V) = -\frac{1}{3}\xi(V)$$

which implies $\xi(V) = 0$ and hence by the previous equation, $V = 0$. This shows ellipticity.

Finally, if S is any traceless symmetric two-tensor and Z a vector field, then we have

$$(Z, -div_g(S))_{L^2} = (\nabla Z, S)_{L^2} = \frac{1}{2}\,(L_g Z, S)_{L^2}\,.$$

It is in the last equation where we use the symmetry and trace-free properties. So we see that if Z is a CKV, then it is L^2-orthogonal to the right-hand side of the PDE for W above. By the Hodge decomposition/Fredholm alternative, we can thus solve the PDE, uniquely up to the kernel. ■

Now let g_0 be a given metric, and g a metric conformally related to g_0, by $g = \phi^4 g_0$. We recall the scalar curvature formula

$$R(g) = \phi^{-5}(-8\Delta_{g_0}\phi + R(g_0)\phi).$$

The following formula is a straightforward (maybe laborious) calculation: for any symmetric, trace-free (2,0)-tensor Ξ (note that it's trace-free with respect to both g and g_0 then), we have

$$div_g(\phi^{-10}\Xi) = \phi^{-10}(div_{g_0}\Xi).$$

We also have for any vector field W

$$(L_g W)^{ab} = \phi^{-4}\,(L_{g_0} W)^{ab}$$

where the term on the right-hand side has been raised by g_0; a consequence is the obvious fact that the CKV's are the same for each metric.

Therefore, given Ξ a symmetric (2,0)-tensor, with its TT-decomposition with respect to g_0 written (raise the TT-decomposition for Ξ_{ab})

$$\Xi^{ab} = \Xi^{ab}_{TT} + \Xi^{ab}_{L} + \Xi^{ab}_{Tr}$$

we have that

$$\phi^{-10}\Xi^{ab} = \phi^{-10}\Xi^{ab}_{TT} + \phi^{-10}\Xi^{ab}_{L} + \phi^{-10}\Xi^{ab}_{Tr}.$$

Now it's easy to verify the trace parts are preserved (as the first two terms are trace-free), and then the previous formula about the divergence shows that $\phi^{-10}\Xi^{ab}_{TT}$ is TT with respect to g.

Finally we arrive at the conformal method. Instead of specifying g and K, we'll instead specify a metric g_0 (to be in the conformal class of g) and the trace- and TT-parts of the symmetric tensor K at g_0, namely the trace function τ and the symmetric TT-tensor σ^{cd}. Given this data, we search for a solution in the conformal class of this data, namely, we pick a positive function ϕ and construct g and K as

$$\begin{aligned} g &= \phi^4 g_0 \\ K^{cd} &= \phi^{-10}\left(\sigma^{cd} + (L_{g_0}W)^{cd}\right) + \frac{1}{3}\phi^{-4}g_0^{cd}\tau. \end{aligned}$$

Note that $g^{cd} = \phi^{-4}g_0^{cd}$, so that $g_{cd}K^{cd} = \tau$. With $K_{ab} = g_{ac}g_{bd}K^{cd}$, we plug g and K into the constraint equations and compute the following system (using the remarks in the previous paragraph):

$$\begin{aligned} \Delta_{g_0}\phi &= \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}\left|\sigma + L_{g_0}W\right|^2_{g_0} + \frac{1}{12}\tau^2\phi^5 \\ div_{g_0}\left(L_{g_0}W\right) &= \frac{2}{3}\phi^6\nabla_{g_0}\tau. \end{aligned} \tag{2}$$

So the problem now is to determine for which (g_0, σ, τ) can we solve for $\phi > 0$, W satisfying the above system, which is an elliptic system of four equations for four unknowns.

Remark. First, we remark that given a solution (g, K) of the constraints, conformal data which gives rise to this solution is just (g, σ, τ), where σ is the TT-part of K, and τ is the trace of K. Also note that any conformal Killing field can be added to a solution W, but this will not change the "real" data K. ■

To simplify matters, we now restrict to the CMC case (τ constant), for which the second equation admits the solution $W = 0$. So then we are just looking at the Lichnerowicz equation

$$\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} + \frac{1}{12}\tau^2\phi^5.$$

Remark. Conformal invariance in the CMC case is expressed as follows: if (g_0, σ, τ) admits a solution ϕ, then $(\theta^4 g_0, \theta^{-10}\sigma^{cd}, \tau)$ admits a solution $\phi\theta^{-1}$ giving rise to the *same* physical data (g, K).

Theorem (Choquet-Bruhat, O'Murchadha, York, Isenberg). On an oriented, closed Σ^3, let (g_0, σ^{cd}, τ) be CMC conformal data (τ constant). The Lichnerowicz equation admits a *positive* solution ϕ as

indicated in the table below; "$\mathcal{Y}$" indicates the Yamabe class of g_0. The solution is unique except in the case $(\mathcal{Y}^0, \sigma = 0, \tau = 0)$ in which case any positive constant is a solution. If g_0 is C^3, σ is $W^{2,p}$ $(p > 3)$, then ϕ is in $C^{2,\alpha}$, and if everything is smooth, then ϕ is smooth.

	$\sigma = 0,\ \tau = 0$	$\sigma = 0,\ \tau \neq 0$	$\sigma \neq 0,\ \tau = 0$	$\sigma \neq 0,\ \tau \neq 0$
$\mathcal{Y}^+$	No	No	Yes	Yes
$\mathcal{Y}^0$	Yes	No	No	Yes
$\mathcal{Y}^-$	No	Yes	No	Yes

In particular, since we showed that any closed manifold admits metrics in $\mathcal{Y}^-$, we see that any closed oriented Σ^3 admits families of solutions to the constraints.

Proof: First of all, by the conformal invariance, we can without loss of generality choose any metric we want in the given conformal class to be g_0; by the Yamabe Theorem, we can then choose the scalar curvature of the conformal representative g_0 to be *constant*. We remark that the non-existence results follow from the maximum principle, and the existence results can be established via super- and sub-solutions. We illustrate with a few cases below, and refer the other cases to Isenberg's paper [I]. We let $\sigma^2 = |\sigma|^2_{g_0}$.

Case : $g_0 \in \mathcal{Y}^0$, $\sigma \neq 0$, $\tau = 0$.

Without loss of generality, we can take $R(g_0) \equiv 0$. The Lichnerowicz equation is then

$$\Delta_{g_0}\phi = -\frac{1}{8}\sigma^2\phi^{-7}$$

where the right-hand side is not identically zero. A solution $\phi > 0$ would be a superharmonic function $(\Delta_{g_0}\phi \leq 0)$, so by the Strong Maximum Principle [G-T], ϕ must be constant. But this cannot be since σ is nontrivial.

Case : $g_0 \in \mathcal{Y}^+$, $\sigma = 0$, $\tau \neq 0$.

In this case we can take $R(g_0) \equiv 1$, so the equation then becomes

$$\Delta_{g_0}\phi = \frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5.$$

A positive solution will satisfy $\Delta_{g_0}\phi > 0$, which by the Maximum Principle cannot happen.

Now we discuss several of the existence cases:

Case : $g_0 \in \mathcal{Y}^0$, $\sigma = 0$, $\tau = 0$.

With $R(g_0) \equiv 0$, so that the equation is just $\Delta_{g_0}\phi = 0$. The solutions are constants (we want the positive ones); note this is the only case where there is non-uniqueness.

Case : $g_0 \in \mathcal{Y}^-$, $\sigma = 0$, $\tau \neq 0$.

With $R(g_0) \equiv -1$, we have

$$\Delta_{g_0}\phi = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5.$$

It is easy to verify the constant $\phi = \left(\frac{3}{2}\frac{1}{\tau^2}\right)^{\frac{1}{4}}$ is a solution.

Case : $g_0 \in \mathcal{Y}^-$, $\sigma \neq 0$, $\tau \neq 0$.

With $R(g_0) \equiv -1$, we get

$$\Delta_{g_0}\phi = -\frac{1}{8}\phi - \frac{1}{8}\phi^{-7}\sigma^2 + \frac{1}{12}\tau^2\phi^5 := G(x,\phi).$$

It is easy to verify that $\phi_- \equiv \left(\frac{3}{2}\frac{1}{\tau^2}\right)^{\frac{1}{4}}$ is a subsolution:

$$\Delta_{g_0}\phi_- = 0 \geq G(x,\phi_-) = -\frac{1}{8}\phi_-^{-7}\sigma^2.$$

We construct a positive supersolution ϕ_+ by finding a constant $\phi_+ \geq \phi_-$ with $G(x,\phi_+) \geq 0$, *i.e.*

$$\frac{2}{3}\tau^2\phi_+^5 \geq \phi_+ + \sigma^2\phi_+^{-7}.$$

We can let

$$\phi_+ = \max\left[1, \left(\frac{3}{2}\frac{1}{\tau^2}(1+\max\sigma^2)\right)^{\frac{1}{4}}\right].$$

Case : $g_0 \in \mathcal{Y}^+$, $\sigma \neq 0$, $\tau = 0$.

Take $R(g_0) \equiv 8$, so the equation is

$$\Delta_{g_0}\phi = \phi - \frac{1}{8}\sigma^2\phi^{-7}.$$

We will again construct super- and sub-solutions, but unlike above, the subsolution will not be constant. Indeed, let $A = \max\left(1, \frac{1}{8}\max\sigma^2\right)$. Let ϕ_- be a solution to the equation

$$-\Delta_{g_0}\phi_- + \phi_- = \frac{\sigma^2}{8}A^{-7}.$$

As the operator $(-\Delta_{g_0}+1)$ is self-adjoint with no kernel, we can uniquely solve the above equation; we note that in the smooth case, ϕ_- will be smooth, and in the regularity mentioned in the theorem, we can get $\phi_- \in W^{4,p}$. By the Maximum Principle, ϕ_- cannot have a negative minimum: at the minimum point p, then $\Delta_{g_0}\phi_- \geq 0$, which would violate the PDE for ϕ_-. So then by the Strong Maximum Principle, we see that $\phi_- > 0$.

We now show that ϕ_- is a subsolution. Let $F(x,s) = \frac{\sigma^2}{8}s^{-7}$, so that F is non-increasing in the s-variable. Hence, since $A \geq 1$, we have

$$F(x,A) \leq F(x,1) = \frac{\sigma^2}{8} \leq A.$$

We then have

$$-\Delta_{g_0}\phi_- + \phi_- = F(x,A) \leq A.$$

By the Maximum Principle, then, $\phi_- \leq A$, so that $F(x,A) \leq F(x,\phi_-)$, and this implies that ϕ_- is a subsolution of the Lichnerowicz equation in this case.

For a supersolution, let $\phi_+ \equiv A \geq \phi_-$. Then we get

$$-\Delta_{g_0}\phi_+ + \phi_+ = \phi_+ = A \geq F(x, A) = F(x, \phi_+). \qquad \blacksquare$$

Remark. We remark that in a subsequent paper, Isenberg and Moncrief extend the results from the CMC case to the almost-CMC case, in which there is an appropriate smallness condition on $\nabla\tau$ [I-M].

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