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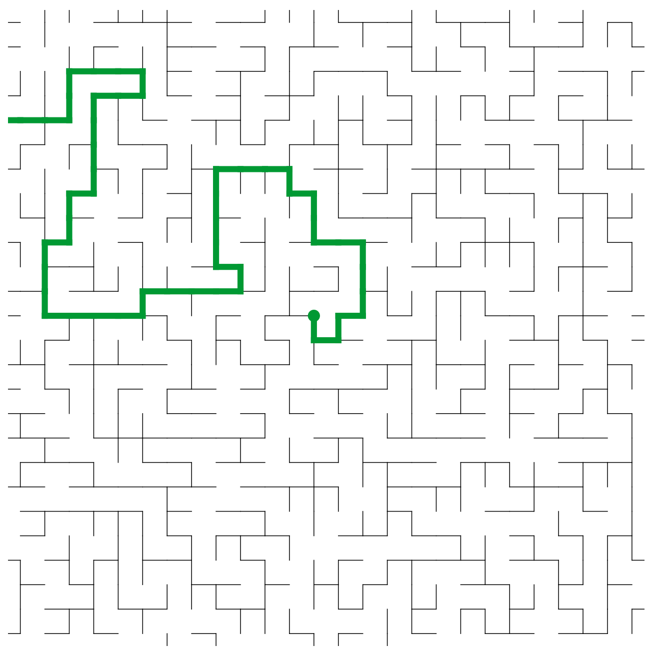
title: The W-conjecture for loop-erased random walk

problem:

Consider a large portion of the square grid, say $[-n, n] \times [-n, n]$, say with wired boundary conditions. Consider a uniformly random spanning tree on this grid. There is some path within the tree connecting $(0, 0)$ to the boundary. This path is a loop-erased random walk (LERW) from $(0, 0)$ stopped upon hitting the boundary. Upon taking the limit $n \to \infty$, this path converges in distribution to what is known as LERW on $\mathbb{Z}^2$. The figure depicts a portion of a random spanning tree on $\mathbb{Z}^2$ together with the path (shown in green) from $(0, 0)$ to ∞, which is the LERW on $\mathbb{Z}^2$.



By symmetry, the LERW path from $(0, 0)$ to ∞ includes the directed edge $(0, 0) \to (1, 0)$ with probability $1/4$, and includes the directed edge $(1, 0) \to (0, 0)$ with probability 0, since otherwise it would have a loop, so altogether the undirected edge $(0, 0) \sim (1, 0)$ is in the path with probability $1/4$.

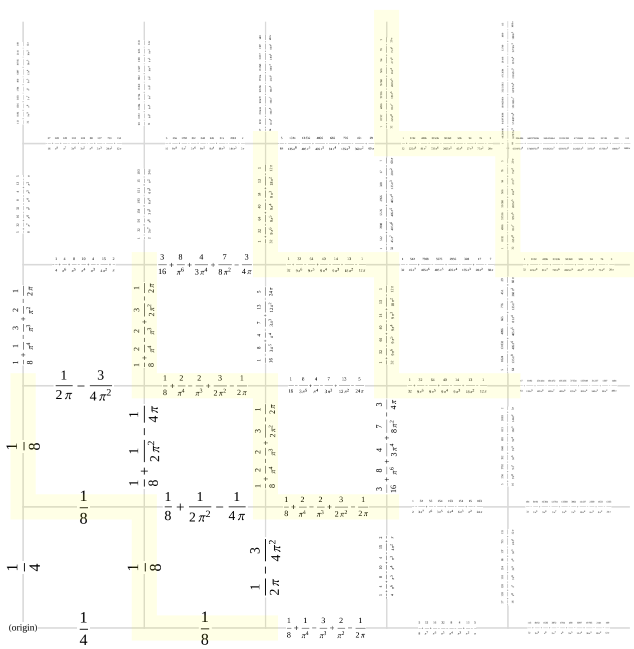
It turns out the the LERW path includes the undirected edges $(1, 0) \sim (1, 1)$ and $(1, 0) \sim (2, 0)$ each with probability $1/8$. There is no lattice symmetry between these two edges, so this could be regarded as a minor coincidence. Of course, by reflecting along the diagonal line passing through the origin, we see that the edges $(1, 1) \sim (0, 1)$ and $(0, 1) \sim (0, 2)$ also occur

within the LERW with probability 1/8. These four edges collectively form a W-shape rotated 45°.

In fact, there are more W's of edges that occur with equal probability. For example, the W passing through $(2,2)$ contains the edges $(2,2) \sim (2,1)$, $(2,1) \sim (3,1)$, $(2,2) \sim (1,2)$, and $(1,2) \sim (1,3)$, all of which occur within the LERW from $(0,0)$ to ∞ with probability

$$\frac{1}{8} - \frac{1}{2\pi} + \frac{3}{2\pi^2} - \frac{2}{\pi^3} + \frac{2}{\pi^4}.$$

The W's passing through $(3,3)$ and $(4,4)$ also have edges that occur with equal probability. The undirected edge probabilities in the LERW are shown in the figure, together with these W's highlighted in yellow.



Conjecture: For any $n > 0$, the edges $(n,n) \sim (n,n-1)$ and $(n,n-1) \sim (n+1,n-1)$ occur with equal probability within the loop-erased random walk from $(0,0)$ to ∞.

background:

Lawler introduced loop-erased random walk and showed that on $\mathbb{Z}^2$ it is well-defined. Pemantle showed the connection between LERW and random spanning trees, and showed that random spanning trees on $\mathbb{Z}^2$ are well-defined. For modern proofs and additional results on random spanning trees, see the article by Benjamini, Lyons, Peres, and Schramm, "Uniform spanning forests" *Ann. Probab.*, 29(1):1–65, 2001. The above LERW edge probabilities were computed by Kenyon and Wilson in `http://arxiv.org/abs/1107.3377`.