Introduction to General Relativity

MSRI Summer Graduate Workshop on

Mathematical General Relativity

July, 2012

Postulates of General Relativity

- A *spacetime* is a connected, time-oriented, Lorentz manifold.
- *Free fall*, motion solely under the influence of gravity, is geodesic.
- *Einstein equation* is satisfied.

A *Lorentz metric* $\bar{g}$ on a smooth manifold M is a symmetric, nondegenerate $(0,2)$ tensor field of index 1.

We often use $\langle\ ,\ \rangle$ as an alternative notation for $\bar{g}$.

A tangent vector $V \in TM$ is called

$$\begin{cases} \textit{spacelike} & \textit{if} \quad \langle V, V\rangle > 0 \textit{ or } V = 0 \\ \textit{timlike} & \textit{if} \quad \langle V, V\rangle < 0 \\ \textit{null} & \textit{if} \quad \langle V, V\rangle = 0 \ (\textit{and } V \neq 0) \end{cases}$$

At each $p \in M$, the set of all timelike vectors consists of two disjoint, open cones in T_pM, called the timecones.

$(M, \bar{g})$ is said to be *time-orientable* if there exists a continuous way of assigning to each point $p \in M$ a timecone $\tau_p \subset T_p M$. That is, for each $q \in M$, there is a continuous timelike vector field V on some open neighborhood U of q such that

$$p \in U \Rightarrow V_p \in \tau_p.$$

Such a function τ is called a *time-orientation* of $(M, \bar{g})$.

Exercise *A Lorentz manifold $(M, \bar{g})$ is time-orientable if and only if there exists a smooth timelike vector field on M.*

One knows every smooth manifold M admits a Riemannian metric. What about Lorentz metrics?

Proposition

A smooth manifold M admits a time-orientable Lorentz metric if and only if there exists a nowhere vanishing vector field on M.

<u>Exercise</u> *Construct a Lorentz metric that is not time-orientable.*

Definition A *spacetime* $(M, \bar{g})$ is a connected, time-oriented, Lorentz manifold.

An *observer* in a spacetime $(M, \bar{g})$ is defined to be a *future-directed timelike curve* $\gamma : I \to M$ with $|\gamma'(\tau)| = 1$, $\forall \tau \in I$. The parameter τ is called the *proper time* of γ.

An *instantaneous observer* is simply defined to be a future-directed timelike unit tangent vector.

An observer γ is said to be *freely falling* if γ is a geodesic.

Physically, "free falling" means motion solely under influence of gravity. Mathematically, one wants to use geodesics to understand "gravity". This is done by studying how a freely falling observer gets observed by nearby other freely falling observers.

Let $\gamma = \gamma(t) : [0, b] \to M$ be a (timelike) geodesic. Let

$$x = x(\cdot, \cdot) : (-\epsilon, \epsilon) \times [0, b] \to M$$

be a geodesic variation of γ, i.e.

- x is a smooth map with $x(0, t) = \gamma(t)$
- the curve $x_s(\cdot) = x(s, \cdot)$ is a geodesic $\forall\ s \in (-\epsilon, \epsilon)$.

Let $V(t) = \frac{\partial x}{\partial s}|_{s=0}$ be the variation vector field along γ. Then

$$\{x_s\} \textit{ are geodesics } \Rightarrow V'' + R(V, \gamma')\gamma' = 0. \tag{1}$$

Remark: The curvature convention used here is
$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$

Equation (1) has a vivid physical interpretation. One re-write it as

$$V'' = -R(V, \gamma')\gamma'. \tag{2}$$

For $|\Delta s| \ll 1$, $\Delta sV = \Delta s \frac{\partial x}{\partial s}|_{s=0}$ represents the approximate position of a nearby observer $x_{\Delta s}$ **relative to** $x_0 = \gamma$. According to the instantaneous observer $u = \gamma'$, ΔsV obeys the equation

$$(\Delta sV)'' = -R(\Delta sV, u)u. \tag{3}$$

If u still lives in the Newtonian world, u would interprete the motion of ΔsV using Newton's 2nd Law

$$F = ma, \tag{4}$$

where the acceleration a (relative to u) is $(\Delta sV)''$.

Taking the parameter m (known as the rest mass of $x_{\Delta s}$) to be 1, one arrives at the following analogy

$$\text{``} -R(\Delta s V, u)u\text{''} \;\sim\; \text{a kind of force exerted on } x_{\Delta s}.$$

Definition *Let $u \in T_pM$ be a tangent vector to a spacetime $(M, \bar{g})$. The map*

$$F_u(\cdot) : T_pM \longrightarrow T_pM, \text{ where } F_u(v) = -R(v, u)u$$

is called the tidal force (exerted on v) measured by u.

Recall that $\{x_s\}$ are freely falling, i.e. moving solely under the influence of gravity. Therefore, the equation $V'' = -R(V, \gamma')\gamma'$ provides a natural way to come up with the link

$$\text{effect of gravity} \leftrightarrow \text{tidal force} \leftrightarrow \text{the curvature of spacetime.}$$

A a vector field V along a geodesic γ is called a *Jacobi field* if

$$V'' + R(V, \gamma')\gamma' = 0.$$

We will talk more about Jacobi fields later when we discuss Penrose singularity theorem.

In Newtonian gravity, the gravitational force on $\mathbb{R}^n$ takes the form

$$F_{gravity} = -\nabla\phi,$$

where ϕ is the gravitational potential function satisfying the Poisson equation

$$\Delta\phi = 4\pi\rho. \tag{5}$$

Here $\rho \geq 0$ is the *mass density* function on $\mathbb{R}^n$ (often assumed to have compact support).

We want to use (5) to motivate the Einstein Equation.

Let $\{x_s\}$ be a 1-parameter family of freely falling observers considered in Newtonian gravity, that is

$$x = x(\cdot,\cdot) : (-\epsilon, \epsilon) \to \mathbb{R}^n$$

is a smooth map such that each curve $x_s(\cdot) = x(s,\cdot)$ satisfies the equation

$$\frac{d^2 x_s}{dt^2} = F_{gravity}(x_s) = -\nabla\phi(x_s).$$

Taking the s-derivative, one has

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial x}{\partial s}\right) = -\nabla^2\phi \cdot \frac{\partial x}{\partial s} \tag{6}$$

where $\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)_{n\times n}$ is the Hessian of ϕ.

Thus the corresponding “tidal force” in the Newtonian setting is

$$F : \mathbb{R}^n \to \mathbb{R}^n, \text{ where } F(V) = -\nabla^2\phi \cdot V.$$

In terms of F, the Poisson equation (5) takes the form

$$\Delta\phi = \mathrm{trace}(\nabla^2\phi) = \mathrm{trace}(-F(\cdot)) = 4\pi\rho. \tag{7}$$

This would suggest, in GR, a natural analogue of (7) take the form of

$$\mathrm{trace}(-F_u(\cdot)) = \rho(u) \tag{8}$$

where $u \in TM$ is any instantaneous observer and $\rho(u)$ is some quantity representing the mass/energy density observed by u.

On the differential geometry side,

$$\operatorname{trace}(-F_u(\cdot)) = \operatorname{trace}(v \mapsto R(v,u)u) = \operatorname{Ric}(u,u) \tag{9}$$

where $\operatorname{Ric}(\cdot,\cdot)$ is the Ricci curvature of $(M, \bar{g})$ defined by

$$\operatorname{Ric}(u,w) = \operatorname{trace}(v \mapsto R(v,u)w).$$

On the physics side, in GR there is a symmetric $(0,2)$ tensor field $T(\cdot,\cdot)$, called the *stress-energy* tensor, satisfying

$$T(u,u) = \text{the energy density of matter distribution observed by } u. \tag{10}$$

and the conservation Law requires that $T(\cdot,\cdot)$ is divergence free, i.e.

$$\operatorname{div} T = 0. \tag{11}$$

Putting (7), (8), (9) and (10) together, one would expect an equation that the spacetime $(M, \bar{g})$ should satisfy be

$$\mathrm{Ric}(u, u) = \kappa T(u, u), \tag{12}$$

where κ is some normalizing constant. Since both Ric and T are symmetric, a more general form of (12) would look like

$$\mathrm{Ric}(u, v) = \kappa T(u, v) \;\; \forall\, u, v \in TM. \tag{13}$$

However, (13) is not consistent with (11) because

$$\mathrm{div}\ \mathrm{Ric} = \frac{1}{2} dR \tag{14}$$

where $R = \mathrm{trace}(\mathrm{Ric}(\cdot, \cdot))$ is the scalar curvature of $\bar{g}$, which is not necessarily zero.

A direct remedy of (13) using (14) then leads us to the famous Einstein Equation:

$$\mathrm{Ric} - \frac{1}{2}R\bar{g} = \kappa T. \tag{15}$$

Remark:

1) A modified equation $\mathrm{Ric} - \frac{1}{2}R\bar{g} + \Lambda\bar{g} = \kappa T$ is known as the Einstein equation with a cosmological constant Λ.

2) $G := \mathrm{Ric} - \frac{1}{2}R\bar{g} + \Lambda\bar{g}$ is called the *Einstein tensor* of $(M, \bar{g})$.

3) The Einstein Equation $G = \kappa T$ can also be derived from a variational consideration.

Now consider a metric $\bar{g}$ on $\mathbb{R}^4 = \{(t = x^0, x^1, x^2, x^3)\}$ given by

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1.$$

We also assume that $\frac{\partial}{\partial t}$ nearly generates an isometry of $\bar{g}$ as in the Minkowski case, that is $\bar{g}_{\mu\nu,0} \approx 0$.

<u>Exercise</u>: $\mathrm{Ric}_{00} \approx -\frac{1}{2}\Delta_{\mathbb{R}^3}(h_{00})$.

The stress-energy tensor T in this weak field case is often taken to be a dust model, i.e.

$$T = \rho dt \otimes dt.$$

The Einstein equation yields

$$\Delta_{\mathbb{R}^3}\left(-\frac{1}{2}h_{00}\right) \approx \frac{1}{2}\kappa\rho. \tag{16}$$

To identify the role of $-\dfrac{1}{2}h_{00}$, one considers a slowly moving observer $\gamma(\tau) = (t(\tau), x^1(\tau), x^2(\tau), x^3(\tau))$, *with* $|\frac{dx^i}{d\tau}| \ll \frac{dt}{d\tau}$. The geodesic equation of γ shows $\dfrac{d^2x^i}{dt^2} \approx -\Gamma^i_{00} = \dfrac{1}{2}h_{00,i}$, i.e.

$$\frac{d^2\vec{x}}{dt^2} \approx -\nabla_{\mathbb{R}^3}\left(-\frac{1}{2}h_{00}\right),\ \vec{x} = (x^1, x^2, x^3).$$

Thus $-\frac{1}{2}h_{00}$ plays the role of the gravitational potential ϕ in the Newtonian setting. Comparing (16) with (5), one takes $\kappa = 8\pi$.