Causality in a Spacetime III

MSRI Summer Graduate Workshop on

Mathematical General Relativity

July, 2012

Given a causal curve segment β, let $L(\beta)$ denote the length of β.

Basic Lemma *Let $\{\alpha_k|_{[0,1]}\}$ be a sequence of future-directed causal curve segements contained in a compact set K with*

$$\alpha_k(0) \to p \ \ \textit{and} \ \ \alpha_k(1) \to q \neq p.$$

If the strong causality condition holds at every point in K, then there exists a future-directed causal curve $\lambda|_{[0,1]}$ from p to q, which is a piecewise smooth geodesic, such that

$$L(\lambda) \geq \liminf_{m\to\infty} L(\alpha_k).$$

In a spacetime M, the analogue of the Riemannian distance is:

$$\tau(p,q) = \sup\{L(\alpha) \mid \alpha \text{ is a future causal curve from } p \text{ to } q\}.$$

Thus

- $q \notin J^+(p) \Rightarrow \tau(p,q) = 0$
- $q \in I^+(p) \Rightarrow \tau(p,q) > 0$
- $q \in J^+(p) \setminus I^+(p) \Rightarrow \tau(p,q) = 0.$

Hence $\tau(p,q) > 0 \Leftrightarrow q \in I^+(p)$.

<u>Exercise</u> $\tau(\cdot,\cdot) : M \times M \to [0,\infty]$ *is lower semi-continuous.*

Given $p < q$, it is natural to consider the set

$$J(p,q) := J^+(p) \cap J^-(q).$$

Proposition *Given $p < q$, suppose*

- $J(p,q) = J^+(p) \cap J^-(q)$ *is compact*
- *the strong causality condition holds at every point in* $J(p,q)$.

Then there exists a smooth future-directed causal geodesic λ *from p to q with $L(\lambda) = \tau(p,q)$.*

An open set H in M is called a *globally hyperbolic open set* if

- the strong causality condition holds at any $p \in H$
- If $p, q \in H$ and $p < q$, then $J(p, q)$ is compact and is contained in H.

The spacetime M is called *globally hyperbolic* if the above two conditions holds with H replaced by M.

Lemma *If $H \subset M$ is a globally hyperbolic open set, then the relation $\leq$ is closed on H, that is*

$$\text{"}\{p_k\}, \{q_k\} \subset U,\ p_k \leq q_k, p_k \to p \in U,\ q_k \to q \in U\text{"} \quad \Rightarrow p \leq q.$$

Therefore, $J^+(p)$, $J^-(p)$ are always closed sets in a globally hyperbolic spacetime.

$A \subset M$ is achronal. The *future domain of dependence* of A is

$D^+(A) = \{p \mid \text{every past inextendible causal curve from } p \text{ meets } A\}$

Clearly, $A \subset D^+(A) \subset (A \cup I^+(A)) \subset J^+(A)$.

The *past domain of dependence* of A is

$D^-(A) = \{p \mid \text{every future inextendible causal curve from } p \text{ meets } A\}$.

The (total) domain of dependence of A is

$$D(A) = D^+(A) \cup D^-(A).$$

Proposition *Let A be an achronal set. Then $\mathrm{int}(D(A))$, if nonempty, is a globally hyperbolic open set in M.*

Ask: for which achronal set A, is $D(A)$ necessarily an open set?

One simple sufficient condition is

"*Suppose A is an achronal set that does not contain edge points. If in addition A is known to be* acausal, *i.e. no causal curve meets A more than once, then $D(A)$ is an open set.* "

An achronal set $S \subset M$ is called a *Cauchy hypersurface* if

$$D(S) = M,$$

that is, S is met by every inextendible causal curve in M.

If S is a Cauchy hypersurface, then

- M is the *disjoint union* of $I^-(S),\ S$ and $I^+(S)$
- S is a closed set and $\mathrm{edge}(S) = \emptyset$. Hence, S is a C^0 hypersurface
- M is globally hyperbolic.

If S is a Cauchy hypersurface, then S is met exactly once by any inextedible timelike curve. This suggests one use timelike vector fields to study S.

Let T be a smooth timelike vector field on M. Let $F(p,t)$ be the maximal integral curve of T passing $p \in M$. Then $F(\cdot,\cdot) : \mathcal{D} \longrightarrow M$ is a smooth map, where $\mathcal{D} \subset M \times \mathbb{R}^1$ is an open set of the form $(\{p\} \times \mathbb{R}^1) \cap \mathcal{D} = \{p\} \times I_p$ where I_p is an open interval.

Suppose $S \subset M$ is a Cauchy hypersurface, then

$$F_p(t) \text{ meets S exactly once}, \quad \forall\ p \in M.$$

Consider $H = F(\cdot,\cdot) : (S \times \mathbb{R}^1) \cap \mathcal{D} \longrightarrow M$. Then $H(\cdot,\cdot)$ is continuous, one-to-one and onto. Thus, H is a homeomorhism by the invariance of domain.

Let $\rho = \pi \circ H^{-1} : M \to S$ where $\pi : S \times \mathbb{R}^1 \to S$ is the projection. Then ρ is a continuous, open and onto map with $\rho|_S = id$.

Corollary *Suppose S is a Cauchy hypersurface in M. Then*

1. *S is necessarily connected.*
2. *If $\tilde{S}$ is another Cauchy hypersurface, then S and $\tilde{S}$ are homeomorphic.*
3. *Suppose A is an achronal, C^0 hypersurface. If A is compact, then $\rho|_A : A \to S$ is a homeomorphism. Consequently, S must be compact.*