More on Schwarzschild Spacetimes

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Consider the Schwarzschild spacetime metric

$$\bar{g}_{\mathfrak{m}} = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 g_{\mathbb{S}^2}, \;\; h(r) = 1 - \frac{2\mathfrak{m}}{r}$$

defined on $P \times S^2$, where

- $\mathfrak{m} < 0$: $P = \mathbb{R} \times \mathbb{R}^+$
- $\mathfrak{m} > 0$: $P = \mathbb{R} \times (0, 2\mathfrak{m}) \cup (2\mathfrak{m}, \infty)$.

It is crucial to understand 2-d spacetime (P, ds^2), where

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2.$$

First, we consider its curvature.

In a general Lorentz manifold M, let $\pi \subset T_pM$ be a 2-d subspace. π is called *non-degenerate* if

$$\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2 \neq 0 \text{ for any basis } \{v, w\} \text{ of } \pi.$$

This simply means that the Lorentz metric on T_pM restricted to π is still non-degenerate.

For a non-degenerate $\pi \subset T_pM$, its the sectional curvature is defined as

$$K(\pi) = \frac{\langle R(v, w)w, v\rangle}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2}$$

where $\{v, w\}$ is any basis for π.

On (P, ds^2), one computes $K = \dfrac{2\mathfrak{m}}{r^3}$, which blows up at $r = 0$, suggesting $r = 0$ is a singularity for (P, ds^2).

Next, we seek all the null geodesics in (P, ds^2).

Fact: $(t, r) \mapsto (t + c, r)$, $(t, r) \mapsto (-t, r)$ are isometries of ds^2.

It suffices to find one null geodesic. Let $\gamma : I \to (P, ds^2)$ be a null geodesic with $\gamma(s) = (t(s), r(s))$:

- $\langle \gamma', \gamma' \rangle = 0 \Rightarrow -h(r)t'(s)^2 + h(r)^{-1}r'(s)^2 = 0$
- $\nabla_{\gamma'}\gamma' = 0$ *and* ∂_t *is Killing* $\Rightarrow \langle \gamma', \partial_t \rangle = \mathit{constant}$, i.e.

$$-h(r)t'(s) = \mathit{constant}$$

Thus, $r'(s)^2 = h(r)^2 t'(s)^2 = \mathit{constant}$. Let's take $r'(s) = 1$. Then, $t'(s)^2 = h(r)^{-2}$. Taking $t'(s) = h(r)^{-1}$, one sees

$$\gamma(s) = (t(s), r(s)) = (s + 2\mathfrak{m} \ln|s - 2\mathfrak{m}|, s)$$

is a null geodesics.

Composing this null geodesic with the known isometries, we know

$$\gamma(s) = (\pm[s + 2\mathfrak{m}\ln|s - 2\mathfrak{m}|] + A, s),$$

where A is a constant, yields all null geodesics in (P, ds^2).

Suppose $\mathfrak{m} < 0$: $\forall\ (t_0, r_0) \in P = \mathbb{R} \times \mathbb{R}^+$, consider the graph of

$$t(r) = r + 2\mathfrak{m}\ln(r - 2\mathfrak{m}) + A$$

passing (t_0, r_0) over $(0, r_0]$, which corresponds to a past-directed null geodesic emanating from (t_0, r_0). This null geodesic terminates at the singularity $r = 0$, after *finite* affine parameter.

What this says is, given any event $p = (t_0, r_0)$, there is a future-directed null geodesics, "starting" from the singularity $r = 0$, that reaches p after *finite* affine parameter.

Suppose $\mathfrak{m} > 0$: in this case we have $P = P_I \cup P_{II}$, where

$$P_I = \mathbb{R} \times (2\mathfrak{m}, \infty) \ \ \textit{and} \ \ P_{II} = \mathbb{R} \times (0, 2\mathfrak{m}).$$

- In (P_I, ds^2): $h(r) > 0$, ∂_t is timelike, ∂_r is spacelike, every null geodesic $\gamma(s) = (\pm[s + 2\mathfrak{m} \ln(s - 2\mathfrak{m})] + A, s)$ is *asymptotic* to the vertical line $r = 2\mathfrak{m}$ as $r \to 2\mathfrak{m}+$.
- In (P_{II}, ds^2): $h(r) < 0$, ∂_t **is spacelike,** ∂_r **is timelike**, every null geodesic $\gamma(s) = (\pm[s + 2\mathfrak{m} \ln|s - 2\mathfrak{m}|] + A, s)$ is *asymptotic* to the vertical line $r = 2\mathfrak{m}$ as $r \to 2\mathfrak{m}-$, and terminates at the t-axis as $r \to 0+$.

If we stick to the (t, r) coordinates, (P_I, ds^2) and (P_{II}, ds^2) seem to bear no relation to each other. To possibly relate them, we rewrite the metric ds^2 in P_I and P_{II} using *null coordinates*:

For convenience, let $r_* = r + 2\mathfrak{m} \ln \left| \frac{r-2\mathfrak{m}}{2\mathfrak{m}} \right|$, define

$$v = t + r_*, \quad and \quad u = t - r_*.$$

The fact $dr_* = \dfrac{1}{h(r)} dr \Rightarrow dudv = dt^2 - \dfrac{1}{h^2(r)} dr^2$. Therefore,

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)} dr^2 = -h(r)dudv.$$

The region $P_I = \{(t, r) \mid -\infty < t < \infty, 2\mathfrak{m} < r < \infty\}$ gets sent to the entire uv-plane, which we denote by $\mathbb{R}^2_{uv}$.

On $\mathbb{R}^2_{uv}$, r becomes a smooth function of u, v determined by

$$\frac{v-u}{2} = r_* = r + 2\mathfrak{m} \ln\left(\frac{r-2\mathfrak{m}}{2\mathfrak{m}}\right). \tag{1}$$

Thus,

$$\frac{2\mathfrak{m}}{r} e^{\frac{v-u}{4\mathfrak{m}}} = e^{\frac{r}{2\mathfrak{m}}}\left(1 - \frac{2\mathfrak{m}}{r}\right) = e^{\frac{r}{2\mathfrak{m}}} h(r).$$

So on $\mathbb{R}^2_{uv}$,

$$ds^2 = -\frac{2\mathfrak{m}}{r} e^{\frac{v-u}{4\mathfrak{m}}} e^{-\frac{r}{2\mathfrak{m}}}\ dudv.$$

The next change of variable is evident:

$$V = e^{\frac{v}{4\mathfrak{m}}} \quad and \quad U = e^{\frac{-u}{4\mathfrak{m}}}$$

with $dV = \frac{1}{4\mathfrak{m}} e^{\frac{v}{4\mathfrak{m}}} dv$, $dU = -\frac{1}{4\mathfrak{m}} e^{\frac{-u}{4\mathfrak{m}}} du$.

Now $\mathbb{R}^2_{uv}$ gets sent to the first quadrant of the UV-plane

$$Q_I = \{(U, V) \mid U > 0,\ V > 0\}.$$

On Q_I, the Schwarzschild metric ds^2 becomes

$$ds^2 = \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} dVdU \tag{2}$$

where $r > 2\mathfrak{m}$ on Q_I is a smooth function of U, V determined by

$$UV = e^{\frac{r}{2\mathfrak{m}}} \left(\frac{r}{2\mathfrak{m}} - 1 \right). \tag{3}$$

Fact: The function $f(r) = e^{\frac{r}{2\mathfrak{m}}} \left(\frac{r}{2\mathfrak{m}} - 1 \right)$ is a smooth function on $(-\infty, \infty)$, and is a *diffeomorphism* from $(0, \infty)$ to $(-1, \infty)$.

Let $f^{-1} : (-1, \infty) \longrightarrow (0, \infty)$ denote the associated smooth inverse function.

The function $r > 2\mathfrak{m}$ on Q_I gets extended to a smooth **positive** function, which we still denoted by r, defined by $r = f^{-1}(UV)$ on the connected open region

$$Q_{UV} = \{UV > -1\} \text{ in the } UV \text{ plane.}$$

Therefore, the map

$$(t, r) \mapsto (V, U) \text{ from } P_I \text{ to } Q_I$$

gives an isometry from (P_I, ds^2) to Q_I in $\left(Q_{UV}, \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} dVdU\right)$.

Tracing back the changes of variables used, we have

$$t = 2\mathfrak{m} \ln \left(\frac{V}{U} \right) \quad \textit{and} \quad r = f^{-1}(UV). \tag{4}$$

Similarly, we can show that (P_{II}, ds^2) is isometric to the region

$$Q_{II} = \{(U, V) \mid U < 0,\ V > 0,\ UV > -1\}$$

in $\left(Q_{UV}, \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} dVdU \right)$ where the isometry is given by

$$t = 2\mathfrak{m} \ln \left| \frac{V}{U} \right| \quad \textit{and} \quad r = f^{-1}(UV). \tag{5}$$

This new 2-d spacetime

$$\left(Q_{UV}, \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} dVdU\right)$$

is called the Kruskal plane. It is easily checked that the map $(U, V) \mapsto (-U, -V)$ is an isometry, and the vector field

$$X = \frac{1}{4\mathfrak{m}}\left(V\partial_V - U\partial_U\right)$$

is a Killing vector field which agrees with ∂_t on each open quadrant of Q_{UV}.

It is also useful to note that, on Q_I, ∂_V and $-\partial_U$ give the two future null directions in (P_I, ds^2).

Often, one makes another change of variable

$$T = \frac{1}{2}(V - U) \;\; and \;\; X = \frac{1}{2}(V + U),$$

the Kruskal plane then takes the form of

$$\left(Q_{XT}, \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} (-dT^2 + dX^2) \right)$$

where

$$Q_{XT} = \{(X, T) \mid X^2 - T^2 > -1\}.$$

The sectional curvature of the Kruskal plane again is given by $K = \frac{2\mathfrak{m}}{r^3}$. Since the curvature blows up at $r = 0$, one concludes that the metric on the Kruskal plane can *not* be smoothly extended across any portion of its hyperbola boundaries.

The Kruskal spacetime is defined to be

$$\left(Q_{UV} \times S^2, \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} dV dU + r^2 g_{\mathbb{S}^2}\right)$$

with another form of

$$\left(Q_{XT} \times S^2, \frac{32\mathfrak{m}^3}{r} e^{-\frac{r}{2\mathfrak{m}}} (-dT^2 + dX^2) + r^2 g_{\mathbb{S}^2}\right).$$