Open Problem Summary - Chris Soteros (soteros@math.usask.ca) from Thurs. Feb. 2, 2012 Problem Session

Problem Statement

Define the $m \times m$ grid graph, $S(m)$, to be the subgraph of the square lattice induced by the vertices $\{(i,j)|0 \leq i \leq m-1, 0 \leq j \leq m-1\}$.

Define $f_{m,k}$ to be the number of spanning forests of $S(m)$ which contain k edges.

Define:

$$F_m(\beta) = \sum_k f_{m,k} e^{\beta k}.$$

Conjecture: $\mathcal{F}(\beta) = \lim_{m\to\infty} \frac{1}{m^2} \log F_m(\beta)$ is analytic for all finite β, that is there is no phase transition for this model.

Related Results

Let $f^r_{m,k}$ be the number of spanning forests on $S(m)$ such that each connected component has a designated root vertex and let

$$F^r_m(\beta) = \sum_k f^r_{m,k} e^{\beta k}.$$

E I Kornilov and V B Priezzhev (1994 Theor. Math. Phys. **98** 61 - see equation (18) with $x = e^\beta$ and $z = 1$) prove that

$$\mathcal{F}^r(\beta) \equiv \lim_{m\to\infty} \frac{1}{m^2} \log F^r_m(\beta) = \beta + \pi^{-2} \int_0^\pi \int_0^\pi \log(4 + e^{-\beta} - 2\cos\theta_1 - 2\cos\theta)2) d\theta_1 d\theta_2$$

which is analytic for all β. This gives an upper bound for $\mathcal{F}(\beta)$.

The partition function $F_m(\beta)$ and the limit $\mathcal{F}(\beta)$ can be related to those of some other problems - these relationships are outlined in detail in L. Stratychuk and C. E. Soteros (1996 J. Phys. A:Math. Gen. **29** 7067) - referred to henceforth as [SS96]. In particular, it is related to: a model of collapsed polymers using spanning connected subgraphs of $S(m)$ (equation (3.13) of [SS96]); the reliability polynomial of $S(m)$ with edge-retention probability $p = \frac{1}{1+e^\beta}$ (equation (3.6) of [SS96]); the Tuttle polynomial $T(S(m); 1, 1+e^{-\beta})$ (equation (3.10) of [SS96]); and, for fixed $p = \frac{1}{1+e^\beta}$, the $q \to 0$ limit of the random-cluster model (see the last paragraph of section 3 of [SS96]).

Further Comments

After the problem session, Lionel Levine suggested looking at "The Random Cluster Model" by G. Grimmett and in particular the section on the pressure. The pressure is introduced on page 86 in Theorem (4.58) and then on page 87 there is some discussion about differentiability. The focus in this section is on $q \in (1,\infty)$ since "the situation $q \in (0,1)$ is obscure". However, from a quick read, it is possible that Conjecture (4.64) on page 87 may have relevance to the conjecture given above?