

Does domino shuffling mix?

a problem presented by James Propp
at Peter Winkler's problem session
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Let T_n denote the set of domino tilings of the Aztec diamond of order $n \geq 0$ (see [arXiv:math/9201305](#) for definitions). Domino shuffling is a (randomized) operation that carries the uniform distribution on T_{n-1} to the uniform distribution on T_n , for all n . In the case $n = 1$, one simply tosses a coin to decide which of the two tilings of the Aztec diamond of order 1 to select. In the case $n = 2$, the two existing tiles (either both horizontal or both vertical) slide apart and create room for four new tiles, whose placement is determined by two coin tosses. When $n \geq 3$, a round of domino shuffling requires performing three steps in succession: annihilation (some of the dominos cancel in pairs), sliding (the surviving dominos slide in various directions determined by their location in the tiling), and creation (vacancies in the tiling are filled by randomly-tiled 2-by-2 blocks). Applying domino-shuffling n times to the trivial probability distribution on T_0 gives a uniformly random domino tiling of the Aztec diamond of order n .

In the original version of domino shuffling, the information that's erased during an annihilation step gets used during the next creation step. Specifically, if k pairs of dominos annihilate, $n + k$ 2-by-2 blocks will need to be tiled, so only n new bits are needed. Here we'll consider a variant in which the $n + k$ blocks are tiled completely randomly (with each of the 2^{n+k} possibilities having probability $1/2^{n+k}$), so that the total number of bits used during the first n stages can exceed $n(n + 1)/2$. Imagine that this process runs forever.

Question 1: Does the mutual information between the tiling at stage m and the tiling at stage n go to 0 if m is fixed and n goes to infinity?

It seems likely that the answer to this question does not depend on m , and in particular that the answer for every m is the same as the answer for $m = 1$. It seems reasonable to hope that an exact formula can be found for the mutual information between the stage-1 tiling and the stage- n tiling.

Question 2: Let P_n^H (resp. P_n^T) denote the conditional distribution on T_n given that the first coin flip (the one used to generate a random tiling in

T_1) was Heads (resp. Tails). Does the total variation distance between P_n^H and P_n^T go to zero?

It's possible that the two questions are equivalent, but I don't see why.

Note that if one uses the original form of domino shuffling (in which no bits are erased), both questions become trivial: the mutual information between the stage- m tiling and the stage- n tiling is $m(m+1)/2$ bits for all $n \geq m$, and the distributions P_n^H and P_n^T are mutually singular. Thanks to Pete Winkler and Zach Hamaker for pointing this out.