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title: Even Orientations and the 3-State Potts Model

problem:

We are interested in sampling weighted even orientations of an $n \times n$ lattice region, also known as the 8-vertex model. Let L be an $n \times n$ region of the Cartesian lattice $\mathbf{Z}^2$, and let Ω_8 be the set of all orientations of the internal edges of L so that all vertices in the interior of L have even in-degree and even out-degree. Given a fixed constant $\lambda \geq 0$ representing the fugacity, for each $\sigma \in \Omega_8$, the Gibbs measure is $\pi(\sigma) = \lambda^{S(\sigma)}/Z$, where $S(\sigma)$ is the number of sources and sinks in σ and Z is the normalizing constant.

Let the Markov chain $\mathcal{M}_8$ be the local chain on Ω_8 that chooses a face of L uniformly at random and either flips the orientations of all internal edges incident to that face or does nothing, according to the correct conditional probabilities. More precisely, let s_1 be the number of sources and sinks among the four vertices defining the face in the current configuration and let s_2 be the number of sources and sinks that would surround that face if we were to flip the orientations of the bounding edges. Then we flip the orientations of the bounding edges with probability $\lambda^{s_2}/(\lambda^{s_1} + \lambda^{s_2})$ and we keep the orientation unchanged with probability $\lambda^{s_1}/(\lambda^{s_1} + \lambda^{s_2})$. This chain is ergodic and converges to π. We are interested in the mixing rate of the chain.

When $\lambda = 0$ the only allowable configurations correspond the 6-vertex model where every vertex has in-degree and out-degree equal to 2. The Markov chain on this part of the state space corresponds to allowing moves only if they reverse a directed cycle around a face, thus maintaining in-degree two at each vertex. This chain is known to be rapidly mixing and corresponds exactly to sampling proper 3-colorings of the region [2, 4, 3]. When $\lambda = 1$, all even orientations are equally likely and the probability of flipping the orientation of edges around any face is the same. If we define the distance between two configurations to be the number of edges in which their orientations differ, then it is easy to construct a coupling argument to show that the chain is rapidly mixing. In fact, when $\lambda = 1$, all moves occur with probability 1/2. We can define a coupling so that the distance function never increases during moves of the coupled chain. When $\lambda \neq 1$ the distance function can increase as well as decrease, but the coupling argument still can be made to work when λ is sufficiently close to 1. However, when λ is large, the Markov chain requires exponential time to converge to equilibrium [1]. This is because if we start from a configuration where all of the even points are sources and all of the odd points are sinks, it will take exponential time to move to a configuration where the majority of vertices have the opposite local configuration.

The first question is to determine whether $\mathcal{M}_8$ is rapidly mixing when $0 < \lambda < .9$.

The picture is almost identical for the 3-state Potts model on L. The state space $\Omega_c = \{0,1,2\}^V(L)$ consists of all colorings of the vertices $V(L)$ of L (not necessarily proper). We are given $\mu \geq 0$ and the stationary probability of a coloring τ is $\pi(\tau) = \mu^{A(\tau)}/Z$ where $A(\tau)$ is the number of edges (u,v) in L such that $\tau(u) = \tau(v)$ and Z is again the normalizing constant. The Markov chain $\mathcal{M}_c$ in this case chooses a vertex at random and recolors that vertex with the correct conditional probabilities. When $\mu = 0$ the chain samples proper

3-colorings and is known to be fast [2, 3]. In fact, when $\mu = 0$ the configurations correspond to the even orientations with $\lambda = 0$ and the two chains are identical. When μ is close to 1 the chain is fast because the color at any vertex is only weakly influenced by any of its four neighbors. And again, when μ is large the chain will be slowly mixing because it will take exponential time to move from a predominantly red configuration to a predominantly blue one. Note that although we see a similar type of behavior, the 8-vertex and 3-coloring models are not identical when $\lambda \neq 0$ and $\mu \neq 0$.

The second question is to determine whether $\mathcal{M}_c$ is rapidly mixing when $0 < \mu < .9$.

References

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