Causality in a Spacetime II

MSRI Summer Graduate Workshop on

Mathematical General Relativity

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M is said to satisfy the *chronology condition* if it contains no timelike loops.

Lemma *The chronology condition never holds on a compact spacetime.*

M is said to satisfy the *causality condition* if it contains no causal loops.

The causality condition is said to hold at $p \in M$ if there is no causal loops passing p.

We say the *strong causality condition* holds at $p \in M$ if given any open neighborhood U of p there is another open neighborhood $V \subset U$ of p such that

$$\alpha|_{[0,1]} \text{ is a causal curve segment, } \alpha(0) \text{ and } \alpha(1) \in V \Rightarrow \alpha|_{[0,1]} \subset U.$$

Lemma *Suppose the strong causality condition holds on a compact subset K. Let α be a future inextendible causal curve that starts in K. Then α eventually leaves K never to return; that is, there is an $t_0 > 0$ such that*

$$\alpha(t) \notin K \;\; \forall \; t \geq t_0.$$

A set $A \subset M$ is called *achronal* if "$p \ll q$" never holds for $p, q \in M$. In other words, A is achronal if $I^+(A) \cap A = \emptyset$.

Fact: A is achronal $\Rightarrow \overline{A}$ is achronal.

Therefore, if $p \in \overline{A} \setminus A$, no timelike curve passing p can meet A.

Given an achronal set A, we say $p \in \overline{A}$ is an *edge point* of A if every open neighborhood U of p contains a timelike curve γ from $I^-(p, U)$ to $I^+(p, U)$ that does not meet A.

Clearly, $\overline{A} \setminus A \subset \mathrm{edge}(A)$.

Proposition

Let A be an achoronal set. Then

$$A \text{ is a } C^0 \text{ hypersurface} \Leftrightarrow A \cap \text{edge}(A) = \emptyset.$$

As a result, A is a closed, C^0 hypersurface $\Leftrightarrow \text{edge}(A) = \emptyset$.

Remark: Here a C^0 hypersurface always means a C^0 hypersurface *without boundary*.

We want to show that, *for any* $A \subset M$, $\partial I^+(A)$ *and* $\partial J^+(A)$ *(if not empty) are always* C^0 *hypersurfaces*.

One way to see this is to first note that $I^+(A)$ and $J^+(A)$ satisfy

$$I^+(I^+(A)) = I^+(A), \quad I^+(J^+(A)) \subset J^+(A).$$

We say a set $F \subset M$ is a future set if $I^+(F) \subset F$.

Proposition

Let F *be a future set in* M*, then* ∂F*, if not empty, is a closed, achronal,* C^0 *hypersurface in* M*.*