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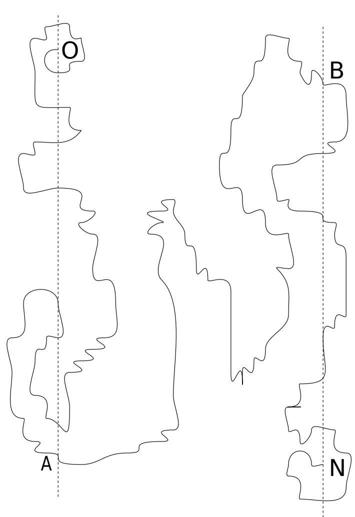
title: An upper bound on the critical exponent for SAWs

problem:

Based on an idea of the late John Hammersley, the following construction provides the basis for a proof that the critical exponent for two-dimensional self-avoiding walks γ, satisfies $\gamma \leq 3$, as usual assuming the existence of such an exponent. The idea relies on the fact that *bridges* are manifestly super-multiplicative, thus if b_n denotes the number of n-step bridges, equivalent up to translation, then $b_n \leq const.\mu^n$, where μ is the growth constant for SAW. (In fact it is believed that $b_n \sim const.\mu^n n^{-\frac{7}{16}}$). Bridges are defined as SAW with a unique left-most origin, with end-point being the (not necessarily unique) rightmost point. Thus they can clearly be concatenated without overlap, hence satisfying $b_n \cdot b_m \leq b_{n+m}$, from which the above result follows (also invoking a trivial lower bound).

We define a new class of SAW called *worms*. These are SAW with the same x co-ordinate for the origin and end point. That is to say, their end-points lie on a line parallel to the y-axis. Experimentally, worms are also super-multiplicative, satisfying $w_n \cdot w_m \leq w_{n+m}$, from which would follow a similar bound. However worms can't be concatenated to produce a new worm (in general), so it is not clear how to establish this inequality. Experimentally, we find $w_n \sim const.\mu^n n^{-\frac{13}{32}}$.



Construction:

In the figure, we have drawn a stylised SAW, and drawn two parallel lattice lines through the origin and the end-pont, labelled O and N respectively. Traversing the walk from the origin, A marks the last time the walk contacts the first lattice line, and B marks the first time it contacts the second. Thus the segment OA is a worm, the segment AB is a bridge, and the segment BN is another worm. If the first segment is of length k, the second of length l, then we can write for the total number of SAW of length n, denoted c_n, the following bound:

$$c_n \leq const.n^2 \cdot \mu^k \cdot \mu^l \cdot \mu^{n-k-l} = const.\mu^n \cdot n^2.$$

The factor n^2 arises from the possible choices of A and B anywhere on the walk. The missing link in what would be a nice result is our inability to prove that worms are super-multiplicative.