Spacetime Singularity Theorems

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What do we mean by saying a spacetime has a "singularity"?

Two explicit examples:

- Robertson-Walker spacetimes with a "big bang"
- "$r = 0$" in a Schwarzschild/Kruskal spacetime with positive mass.

Any reasonable definition of spacetime singularities should at least include these two cases.

Common features shared by the two examples:

- Curvature blows up as an observer approaches the "big bang" or "$r = 0$"
- There exist incomplete causal geodesics "terminating" at the "big bang" or "$r = 0$"

If one uses "*curvature blows up*" as a criteria, some simple examples can be easily missed. For instance, in the Riemannian case, a cone (without vertex) in $\mathbb{R}^n$, which has zero curvature, but the vertex is often considered to be a singularity.

On the other hand, "*incompleteness of certain causal geodesic*" has a clear physical meaning in that it presents the possibility that there is a freely falling observer (or a lightlike particle), along which before or after a finite amount of proper time (or affine parameter), the postulates of GR does not hold.

When using "*incompleteness of certain causal geodesic*" as a criteria, one of course needs to exclude some trivial examples obtained by removing an event from a well-behaved spacetime.

A version of Hawking's Singularity Theorem

(corresponding to the "big bang" in the Robertson-Walker model)

Let M be a spacetime satisfying the following conditions:

1. M has a closed (as a set), spacelike, future Cauchy hypersurface S, meaning $D^+(S) = J^+(S)$, with
$$\theta_S = -\langle \nu, \vec{H} \rangle < -k < 0$$
for some constant $k > 0$.
2. $\mathrm{Ric}(V, V) \geq 0$ for any timelike vector V.

Then "every observer, who is born in S, lives only a finite amount of proper time $\leq \frac{n-1}{k}$." That is to say, every future-directed, inextendible timelike curve has length at most $\frac{n-1}{k}$.

Penrose's Singularity Theorem

Let M be a spacetime satisfying the following conditions:

1. M has a non-compact Cauchy hypersurface
2. $\mathrm{Ric}(V, V) \geq 0$ for all null vectors V.
3. there exists a compact codimension-2 spacelike submanifold Σ such that
$$\theta_\pm := -\langle l_\pm, \vec{H} \rangle < 0 \tag{1}$$
for any future directed null vectors l_+, l_- at any point on Σ. Here $\vec{H}$ is the mean curvature vector of Σ.

Then M is future *null geodesically incomplete*.