

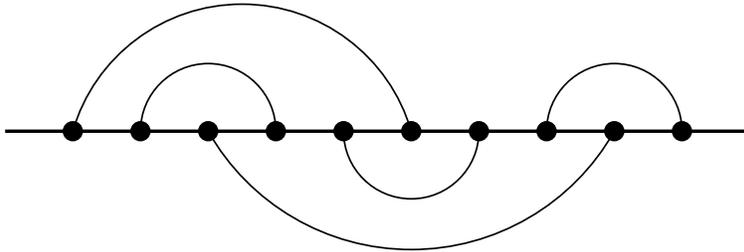
NON-RATIONAL CRITICAL EXPONENTS IN SIMPLE ASYMPTOTIC ENUMERATION PROBLEMS

PHILIPPE DI FRANCESCO

ABSTRACT. The occurrence of non-rational critical exponents tends to be rare in asymptotic combinatorial enumeration problems. Here are a few conjectured examples.

1. THE PROBLEM

1.1. **Preliminaries: the easy case.** Consider the following easy question: *Find the number H_{2n} of configurations of a rooted Hamiltonian cycle on a cubic (trivalent) planar map with $2n$ vertices.* The answer is simple as the counting boils down to that of plane configurations of a straight line through $2n$ vertices (the Hamiltonian cycle cut at the root and then extended), which are connected pairwise via n non-intersecting edges, as in the example below:



The counting boils down to that of a choice of $2m$ vertices among the $2n$ to be connected within the upper half-plane delimited by the line, while the remaining $2n - 2m$ are connected within the lower half-plane. The non-crossing pairwise connections of $2m$ boundary points in a half-plane are enumerated by the Catalan number $c_m = \frac{1}{m+1} \binom{2m}{m}$, so that

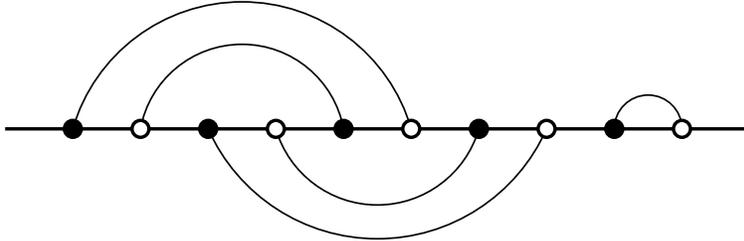
$$H_{2n} = \sum_{m=0}^n \binom{2n}{2m} c_m c_{n-m} = c_n c_{n+1}$$

(The last identity is left as an exercise.). For large n , we have the following asymptotics, from Stirling's formula:

$$(1.1) \quad H_{2n} \sim \frac{4}{\pi} \frac{16^n}{n^3}$$

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1.2. **The hard case.** We now ask the following hard question: *Find the number E_{2n} of configurations of a rooted Hamiltonian cycle on a trivalent Eulerian (i.e., bipartite) planar map with $2n$ vertices.* This boils down to the enumeration of the plane configurations of a line through $2n$ alternating black and white vertices, connected by pairs of opposite colors via non-intersecting edges. Here is a typical example of such a configuration:



The asymptotic answer is conjectured to be [1]:

$$(1.2) \quad E_{2n} \sim C \frac{K^n}{n^\gamma}, \quad \gamma = \frac{13 + \sqrt{13}}{6}$$

for some constants C and $K \simeq 10.104\dots$

2. HINTS AND MORE

2.1. **Critical matter coupled to 2D quantum gravity.** A physicist's view on the problem is to interpret the enumeration problem as that of a matter system (statistical model) with configurations on an underlying fluctuating space, taken from a statistical ensemble of random maps. General wisdom [2] is that if the matter system exhibits some critical behavior on a regular lattice in the thermodynamic limit of large number of sites and small lattice mesh, then it will exhibit some critical behavior on large random maps as well. The relation between both critical behaviors, obtained by field-theoretical arguments, allows to relate critical exponents in both situations. In particular, if the regular lattice model undergoes some phase transition governed by some conformal field theory (CFT) with central charge $c \leq 1$, the critical configuration exponent γ for the same model on a random planar map is given by

$$\gamma = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12}$$

In the case (1.1), the Hamiltonian loop corresponds to the configurations of the dense $O(n = 0)$ loop model, known to be described by a CFT with central charge $c = -2$, which yields $\gamma = 3$. In the case (1.2), the Eulerian character of the map allows for defining a dual height function on the faces of the map, in a way similar to that of dimer covers of the regular hexagonal lattice. The Hamiltonian loop now corresponds to configurations of the fully-packed $O(n = 0)$ loop model, where a single loop covers all the edges not occupied by dimers. The existence of the height function was argued [3] to increase the central charge of the dense model by 1, leading to a central charge $c = -1$, which yields $\gamma = \frac{13 + \sqrt{13}}{6}$.

2.2. Other open problems. The most famous problem in the same class is known as the meander problem: *Find the number M_{2n} of plane configurations of a non-intersecting circuit crossing a river (line) through $2n$ bridges.* Using similar arguments as those briefly described above, it was conjectured that [4]:

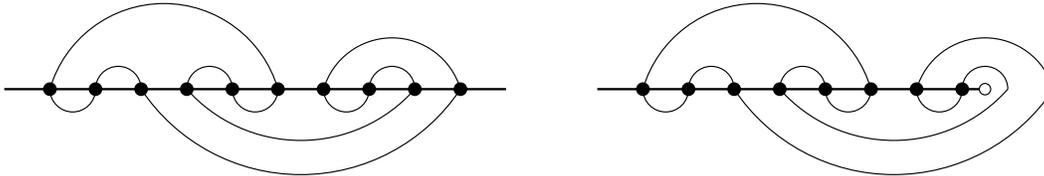
$$M_{2n} \sim \lambda \frac{R^{2n}}{n^\alpha}, \quad \alpha = \frac{29 + \sqrt{145}}{12}$$

for some constants λ and $R \simeq 3.501\dots$. The exponent α corresponds to a CFT with central charge $c = -4$.

Other conjectures involve correlations of observables, such as “river creation operators”, like in the so-called semi-meander problem: *Find the number \bar{M}_n of plane configurations of a non-intersecting circuit crossing a river with a source (half-line) through n bridges.* The conjecture for these reads:

$$\bar{M}_n \sim \mu \frac{R^n}{n^{\bar{\alpha}}}, \quad \bar{\alpha} = 1 + \frac{\sqrt{11}(\sqrt{5} + \sqrt{29})}{24}$$

For illustration, here are a typical meander and semi-meander configurations. The source of the river is indicated by an empty circle:



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INSTITUT DE PHYSIQUE THÉORIQUE DU COMMISSARIAT À L'ÉNERGIE ATOMIQUE, UNITÉ DE RECHERCHE ASSOCIÉE DU CNRS, CEA SACLAY/IPhT/BAT 774, F-91191 GIF SUR YVETTE CEDEX, FRANCE.
E-MAIL: PHILIPPE.DI-FRANCESCO@CEA.FR