On Geometric Riccati and Raychaudhuri Equations

MSRI Summer Graduate Workshop on

Mathematical General Relativity

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M^n: an n-dimensional spacetime

N^k: a spacelike submanifold of dimension $k < n$

$\gamma : [0, L) \to M$, is a causal geodesic with $\gamma(0) = p \in N$ and $\gamma'(0) \perp N$.

Ask: What kind of Jacobi fields V along γ takes account of the pair (γ, N)?

It is natural to consider those Jacobi fields V satisfying

$$V(0) \in T_pN, \tag{1}$$

$$\langle V'(0), W\rangle = -\langle \gamma'(0), \mathrm{I\!I}(V(0), W)\rangle, \ \forall \ W \in T_pN, \tag{2}$$

where $\mathrm{I\!I} : T_pN \times T_pN \to (T_pN)^{\perp}$ is the second fundamental form of N at p.

The set $\tilde{\mathcal{V}}$ of all Jacobi fields V satisfying (1) and (2) is a vector space of dimension $k + (n - k) = n$.

A point $q = \gamma(t)$, $t > 0$, is said to be a focal point of N along γ if there exists a nontrivial element $V \in \tilde{\mathcal{V}}$ such that $V|_q = 0$. If V is such a Jacobi field, it is clear that $V \perp \gamma'$ everywhere.

Therefore, in stead of $\tilde{\mathcal{V}}$, one can focus on

$$\mathcal{V} = \{V \in \tilde{\mathcal{V}} \mid V \perp \gamma'\}$$

whose dimension is $n - 1$.

For each $t \geq 0$, let $\gamma'(t)^{\perp} = \{v \in T_{\gamma(t)}M \mid \langle v, \gamma'(t) \rangle = 0\}$.

Suppose $\gamma(t)$ is *not* a focal point of N along γ for any $t \in (0, L)$:

Then, for each $t \in (0, L)$, the map

$$B = B(t) : \mathcal{V} \to \gamma'(t)^{\perp} \text{ given by } B(t)(V) = V(t)$$

is a linear isomorphism. Now consider

$$A = A(t) : \gamma'(t)^{\perp} \to \gamma'(t)^{\perp} \tag{3}$$

given by

$$A(t)(v) = \left[B(t)^{-1}(v)\right]'(t) \tag{4}$$

where " ′ " denotes covariant differentiation along γ.

By definition, one has

$$A(t)(V(t)) = V'(t), \ \ \forall \ V \in \mathcal{V}. \tag{5}$$

In particular, $\{A(t)\}_{t \in (0,L)}$ is a smooth $(1,1)$ tensor field in the $(n-1)$-dimensional vector bundle $\gamma'(t)^{\perp}$ over γ.

Ask: What is $A'(t) = \nabla_{\gamma'(t)} A(t) : \gamma'(t)^{\perp} \to \gamma'(t)^{\perp}$?

Recall that $\nabla_{\gamma'(t)} A(t)$ is defined by

$$\left[\nabla_{\gamma'(t)} A(t)\right](W(t)) = \nabla_{\gamma'(t)}[A(t)(W(t))] - A(t)[\nabla_{\gamma'(t)} W(t)].$$

for any vector field $W = W(t)$ along γ.

Therefore, given any $V = V(t) \in \mathcal{V}$, we have

$$\begin{aligned} A'(t)(V(t)) =& [A(t)(V(t))]' - A(t)(V'(t)) \\ =& [V'(t)]' - A(t)(A(t)(V(t))) \\ =& -R(V(t), \gamma'(t))\gamma'(t) - A(t) \circ A(t)(V(t)) \end{aligned} \tag{6}$$

where in the last step we used the Jacobi equation. This shows

Proposition

$A = A(t) : \gamma'(t)^\perp \to \gamma'(t)^\perp$ satisfies

$$A' = -R(\cdot, \gamma')\gamma' - A \circ A. \tag{7}$$

Taking trace in $\gamma'(t)^\perp$, one has

$$(\mathrm{tr}_{\gamma'^\perp} A)' = -\mathrm{Ric}(\gamma', \gamma') - \mathrm{tr}_{\gamma'^\perp}(A \circ A). \tag{8}$$

Applications

1) $N = S$ is a spacelike hypersurface:

In this case, γ is necessarily timelike. The spacetime metric restricted to $\gamma'(t)^{\perp}$ is positive definite, still denoted by $\langle \cdot, \cdot \rangle$.

Lemma $A(t) : \gamma'(t)^{\perp} \to \gamma'(t)^{\perp}$ *is self-adjoint w.r.t* $\langle \cdot, \cdot, \rangle$.

Remark The self-adjointness of A does make use of the initial conditions (1) and (2) for V.

Let $h = h(t) : \gamma'(t)^{\perp} \times \gamma'(t)^{\perp} \to \mathbb{R}^1$ be the associated symmetric bilinear form. Define

$$\theta(t) = \mathrm{tr}_{\gamma'(t)^{\perp}} A(t) = \mathrm{tr}_{\gamma'(t)^{\perp}} h(t),$$

then

$$\mathrm{tr}_{\gamma'(t)^{\perp}} [A(t) \circ A(t)] = |h(t)|^2 \geq \frac{1}{n-1} \theta(t)^2.$$

Therefore, we have shown

Proposition *Let S be a spacelike hypersurface in an n-dimensional spacetime. Let γ be a timelike geodesic with*

$$\gamma(0) \in S \text{ and } \gamma'(0) \perp S.$$

Suppose S does not have focal points along γ in $(0, L)$, then there is a well defined smooth function $\theta = \theta(t)$ on $(0, L)$ such that

$$\theta'(t) \leq -\mathrm{Ric}(\gamma', \gamma') - \frac{1}{n-1}\theta(t)^2. \tag{9}$$

Ask: What is $\lim_{t\to 0+} \theta(t)$?

Let $\{V_i \mid i = 1, \ldots, n-1\}$ be a basis for $\mathcal{V}$, let $\sigma_{ij} = \langle V_i, V_j \rangle$. By definition, we have

$$\theta(t) = \sigma^{ij}(t)\langle V_i'(t), V_j(t)\rangle \tag{10}$$

which tends to $\sigma^{ij}(0)\langle V_i'(0), V_j(0)\rangle, \ \textit{as} \ t \to 0+$.

Using initial conditions (1) and (2), we know

$$\langle V_i'(0), V_j(0)\rangle = -\langle \gamma'(0), \mathbb{II}(V_i(0), V_j(0)\rangle. \tag{11}$$

Therefore, we conclude that

$$\theta(0+) := \lim_{t \to 0+} \theta(t) = -\langle \gamma'(0), \vec{H}\rangle \tag{12}$$

where $\vec{H}$ is the mean curvature vector of S at p.

2) $N^k = \Sigma$ is a codimension-2 spacelike submanifold and γ is a null geodesic:

This case is slightly different from the previous case because the spacetime metric restricted to $\gamma'(t)^\perp$ is degenerate. However, this can be easily overcome by considering the quotient space

$$\gamma'^\perp / \sim$$

where $v \sim w$ if $(v - w) \parallel \gamma'$. One easily checks the following

1. the spacetime metric descends to a positive definition metric on $\gamma'^\perp / \sim$, denoted by $\langle \cdot, \cdot \rangle_\sim$.
2. $A : \gamma'^\perp \to \gamma'^\perp$ descends to $\tilde{A} : \gamma'^\perp / \sim \to \gamma'^\perp / \sim$, since $A(t\gamma'(t)) = \gamma'(t)$, and $\tilde{A}$ is self-adjoint w.r.t $\langle \cdot, \cdot \rangle_\sim$.
3. $R(\cdot, \gamma')\gamma' : \gamma'^\perp \to \gamma'^\perp$ descends to $\tilde{R}(\cdot, \gamma')\gamma' : \gamma'^\perp / \sim \to \gamma'^\perp / \sim$, since $R(\gamma', \gamma')\gamma' = 0$.

$\{\tilde{A}(t)\}$ is a smooth $(1,1)$ tensor field in the $(n-2)$-dimensional vector bundle $\gamma'^{\perp}/\sim$ over γ. Let " $'$ " be the corresponding covariant differentiation in this vector bundle. From (7) it follows

$$\tilde{A}' = -\tilde{R}(\cdot, \gamma')\gamma' - \tilde{A} \circ \tilde{A}. \tag{13}$$

Let $\tilde{h} = \tilde{h}(t) : \gamma'(t)^{\perp}/\sim \times \gamma'(t)^{\perp}/\sim \longrightarrow \mathbb{R}^1$ be the associated symmetric bilinear form. Define

$$\theta(t) = \mathrm{tr}_{\gamma'(t)^{\perp}/\sim} \tilde{h}(t),$$

then

$$\mathrm{tr}_{\gamma'(t)^{\perp}/\sim}[\tilde{A}(t) \circ \tilde{A}(t)] = |\tilde{h}(t)|^2 \geq \frac{1}{n-2}\theta(t)^2.$$

Therefore, we have

Proposition *Let Σ be a co-dimension 2 spacelike submanifold in an n-dimensional spacetime. Let γ be a null geodesic with*

$$\gamma(0) \in \Sigma \text{ and } \gamma'(0) \perp \Sigma.$$

Suppose Σ does not have focal points along γ in $(0, L)$, then there is a well defined smooth function $\theta = \theta(t)$ on $(0, L)$ such that

$$\theta'(t) \leq -\mathrm{Ric}(\gamma', \gamma') - \frac{1}{n-2}\theta(t)^2. \tag{14}$$

Moreover,

$$\theta(0+) := \lim_{t \to 0+} \theta(t) = -\langle \gamma'(0), \vec{H} \rangle \tag{15}$$

where $\vec{H}$ is the mean curvature vector of Σ at p.