On the center of mass and foliations of constant mean curvature surfaces in asymptotically flat manifolds

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1 Introduction

This notes is for students in the Introductory Workshop on Mathematical General Relativity at MSRI in July 2012. I will discuss stable constant mean curvature (CMC) surfaces and the center of mass for asymptotically flat manifolds, as well as the density theorems for the Einstein constraint equations, as described below:

(1) Preliminary results on the stable surfaces of CMC, including the variation formulas and the theorem of Barbosa and do Carmo on stable CMC surfaces in Euclidean space.

(2) Existence of CMC surfaces in strongly asymptotically flat manifolds: the volume-preserving mean curvature flow of Huisken and Yau and the perturbation method of R. Ye.

(3) Equivalence of different notions of center of mass. We show that the geometric center of mass given by the family of CMC surfaces is the classical definition of center of mass.

(4) The stability of the CMC surfaces and the local uniqueness.

(5) The density theorem of Corvino and Schoen for the vacuum Einstein constraint equations, and a recent progress on specifying the center of mass and angular momentum for vacuum initial data sets.

2 Definitions

Definition 2.1. *We say that (M^3, g, k) is an asymptotically flat manifold if*

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-q}), \quad k_{ij} = O(|x|^{-1-q}) \quad \textit{and} \quad R(x) = O(|x|^{-3-\epsilon}),$$

where R is the scalar curvature of g, $q > \frac{1}{2}$ and $\epsilon > 0$.

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Definition 2.2. *If (M^3, g, k) is asymptotically flat satisfying the Regge-Teitelboim condition if (M^n, g, k) is asymptotically flat and*

$$g_{ij}(x) - g_{ij}(-x) = O(|x|^{-1-q}), \qquad k_{ij}(x) + k_{ij}(-x) = O(|x|^{-2-q}),$$

and $R(x) - R(-x) = O(|x|^{-4-\epsilon})$, where $q > \frac{n-2}{2}$ and $\epsilon > 0$.

Remark. In most of the theorems about CMC surfaces here, we often require no condition on k, so we denote (M, g) instead of (M, g, k). When we discuss the density theorems, it is convenient to introduce momentum tensor $\pi = k - (\mathrm{tr}_g k)g$.

Definition 2.3. *Let m, C, P, J denote the energy, center of mass, linear momentum, and angular momentum of (g, k). They are defined as limits of integrals over Euclidean spheres*

$$\begin{aligned}
m &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|}\, d\sigma_0,\\
C^p &= \frac{1}{16\pi m} \lim_{r\to\infty} \int_{|x|=r} \left[x^p \sum_{i,j} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|} - \sum_i (g_{ip} \frac{x^i}{|x|} - g_{ii} \frac{x^p}{|x|}) \right] d\sigma_0,\\
P_i &= \frac{1}{8\pi} \lim_{r\to\infty} \int_{|x|=r} \sum_j \pi_{ij} \frac{x^j}{|x|}\, d\sigma_0,\\
J_i &= \frac{1}{8\pi m} \lim_{r\to\infty} \int_{|x|=r} \sum_{j,k} \pi_{jk} Y_i^j \frac{x^k}{|x|}\, d\sigma_0,
\end{aligned}$$

where $d\sigma_0$ is the area measure of the Euclidean sphere $\{|x| = r\}$ and $Y_i = \frac{\partial}{\partial x^i} \times \vec{x}$ (cross product) for $i = 1, 2, 3$ are the rotation vector fields.

3 Basic facts about Schwarzschild solutions

The Schwarzschild solution, which is a static solution to the vacuum Einstein constraints, is the model case for most theorems in general relativity. Before we discuss center of mass and constant mean curvature foliations in general asymptotically flat initial data sets, we first take a closer look at the spatial Schwarzchild solutions. Let

$$g = \left(1 + \frac{m}{2|x - \vec{C}|}\right)^4 \delta = \phi^4 \delta$$

on $\mathbb{R}^3 \setminus \{\vec{C}\}$ be the spatial Schwarzschild solution of mass $m > 0$. Note that g is conformally flat, complete, and scalar flat. Fix m and $\vec{C}$. For $|x| \gg 1$, we consider the leading order terms of the metric

$$g = \left(1 + \frac{2m}{|x|} + \frac{2m\vec{C}\cdot x}{|x|^3} + \frac{3m^2}{2|x|^2} + O(|x|^{-3})\right)\delta.$$

It follows that the mass m appears in the $|x|^{-1}$ of the expansion, while the center of mass appears in the odd part of the $|x|^{-2}$-term.

Using Definition 2.3, we can find that the Schwarzschild solution g has mass m, center of mass $\vec{C}$, zero linear momentum and zero angular momentum. You may notice that the center of mass is not a point inside the manifold, so what does the center of mass mean?

In the case of Schwarzschild solutions, let $S_r = \{\mathbb{R}^3 : |x - \vec{C}| = r\}$ denote the spheres centered at $\vec{C}$. The round spheres centered at $\vec{C}$ form a smooth foliation in the Schwarzschild solution. By rotational symmetry, the round spheres have constant mean curvature. More explicitly, the round spheres S_r are umbilic and have constant mean curvature $\left(2 - \frac{m}{r}\right)\frac{\phi^{-3}}{r}$. It follows that $S_{\frac{m}{2}}$ is the minimal surface, i.e., the horizon of the Schwarzschild solution, and $S_{\frac{(2+\sqrt{3})m}{2}}$ has largest mean curvature and the mean curvature of S_r is increasing in r for $\frac{m}{2} \le r \le \frac{(2+\sqrt{3})m}{2}$ and decreasing in r if $r \ge \frac{(2+\sqrt{3})m}{2}$.

The stability operator $L_{S_r} = -\Delta_{S_r} - (|A_{S_r}|^2 + Ric(\nu,\nu))$ is

$$-\phi^{-4}r^{-2}\Delta_{\mathbb{S}^2} + \frac{-4r^2 + 8rm - m^2}{2r^4\phi^6}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplacian for the standard unit sphere. Hence L_{S_r} and $\Delta_{\mathbb{S}^2}$ have the same eigenfunctions. The lowest eigenvalue of L_{S_r} is

$$\lambda_0 = \frac{-4r^2 + 8rm - m^2}{2r^4\phi^6} = -\frac{2}{r^2} + \frac{10m}{r^3} + O(r^{-4})$$

with the eigenspace space spanned by constants, and the next eigenvalues are

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{6m}{r^3\phi^6} = \frac{6m}{r^3} + O(r^{-4})$$

with the eigenspace spanned by $\{x^1, x^2, x^3\}$. Therefore, $\{S_r\}$ form a foliation of stable surfaces of constant mean curvature in the Schwarzschild solution (see Definition 4.2 on stability).

4 Variation formulas and stable constant mean curvature surfaces in Euclidean space

Let $\Sigma^n \subset M^{n+1}$ be a hypersurface. Let ν be the outward unit normal vector. The mean curvature is defined by $H = \mathrm{div}_\Sigma \nu$. If we parallel translate ν along its normal direction so that ν is defined in an open neighborhood of Σ, then $H = \mathrm{div}_M \nu$. An n-dimensional round sphere of radius r has mean curvature n/r with respect to the outward unit normal in this convention.

We first discuss some variation formulas. Let $\Sigma^n \subset (M^{n+1}, g)$ be a smooth compact hypersurface without boundary. Consider the variation of Σ along its normal direction

$F(\cdot,t) : \Sigma^n \to M^{n+1}$, for $-\epsilon < t < \epsilon$, satisfying

$$\frac{\partial}{\partial t}F(x,t) = \eta(x,t)\nu(x,t)$$
$$F(\Sigma,0) = \Sigma,$$

where $\nu(x,t)$ is the outward-pointing unit normal to $\Sigma_t := \{F(x,t) : x \in \Sigma\}$.

Proposition 4.1. *Let $d\sigma(x,t)$ denote the volume measure of Σ_t. Let A_{Σ_t} and $H(x,t)$ denote the second fundamental form and the mean curvature of Σ_t respectively. Denote the n-volume of Σ_t by $A(t)$ and the $(n+1)$-volume of the region enclosed by Σ_t by $V(t)$. Then*

$$\begin{aligned}
\frac{d}{dt}d\sigma(x,t) &= H(x,t)\eta(x,t)\,d\sigma(x,t),\\
\frac{d}{dt}H(x,t) &= -\Delta_{\Sigma_t}\eta(x,t) - (|A_{\Sigma_t}|^2 + Ric_g(\nu(x,t),\nu(x,t)))\eta(x,t) =: L_{\Sigma_t}\eta(x,t)\\
\frac{d}{dt}\nu(x,t) &= -\nabla^{\Sigma_t}\eta\\
A'(t) &= \int_{\Sigma_t} H(x,t)\eta(x,t)\,d\sigma(x,t) \qquad (4.1)\\
A''(t) &= \int_{\Sigma_t} \eta(x,t)L_{\Sigma_t}\eta(x,t)\,d\sigma(x,t) + \int_{\Sigma_t} H\frac{\partial}{\partial t}\eta(x,t)\,d\sigma(x,t) + \int_{\Sigma_t} H\eta(x,t)\,\frac{\partial}{\partial t}d\sigma(x,t)\\
V'(t) &= \int_{\Sigma_t} \eta(x,t)\,d\sigma(x,t). \qquad (4.2)
\end{aligned}$$

Proof. These formulas are rather standard. Let us, for example, compute the last formula. By the divergence theorem, letting $X(x,t)$ be the position vector, we have

$$(n+1)V(t) = \int_{\Sigma_t} g(X(x,t),\nu(x,t))\,d\sigma(x,t).$$

Hence,

$$\begin{aligned}
(n+1)V'(t) &= \int_{\Sigma_t} \eta(x,t)\,d\sigma(x,t)\\
&\quad - \int_{\Sigma_t} g(X(x,t),\nabla^{\Sigma_t}\eta)\,d\sigma(x,t) + \int_{\Sigma_t} g(X(x,t),\nu(x,t))H(x,t)\eta(x,t)\,d\sigma(x,t)\\
&= (n+1)\int_{\Sigma_t} \eta(x,t)\,d\sigma(x,t),
\end{aligned}$$

where in the last line, we use, for a vector field $X \in TM$ on Σ and $f \in C^1(\Sigma)$,

$$\begin{aligned}
&\int_\Sigma g(\nabla^\Sigma f, X)\,d\sigma + n\int_\Sigma f\,d\sigma = \int_\Sigma \mathrm{div}_\Sigma(fX)\,d\sigma\\
&= \int_\Sigma \mathrm{div}_\Sigma(fX^\perp + fX^t)\,d\sigma = \int_\Sigma fHg(X,\nu)\,d\sigma.
\end{aligned}$$

□

Definition 4.2. *Let $\Sigma^n \subset (M, g)$ be a smooth hypersurface. Denote*

$$\lambda_0 := \left\{ \int_\Sigma \eta L_\Sigma \eta \, d\sigma : \|\eta\|_{L^2(\Sigma)} = 1, \int_\Sigma \eta \, d\sigma = 0 \right\}.$$

Then Σ is called stable if $\lambda_0 \geq 0$; it is called strictly stable if $\lambda_0 > 0$.

Proposition 4.3. *A stable hypersurface Σ with constant mean curvature locally minimizes the area among hypersurfaces which enclose the same volume.*

Proof. Given any η such that $\int_\Sigma \eta \, d\sigma$, there exists a volume-preserving variation $\eta(x,t)$ $V(t) = V(0)$ for all t [1, (2.4) Lemma]. In particular, $V'(0) = 0 = V''(0)$. Then, Σ with constant mean curvature is the critical point of the functional $A(t)$ so that $A'(0) = 0$. Moreover,

$$A''(0) = \int_\Sigma \eta L_\Sigma \eta \, d\sigma + HV''(0) = \int_\Sigma \eta L_\Sigma \eta \, d\sigma \geq 0.$$

Another way to see this is to use the method of Lagragian multipliers and consider the local minimizers of $J(t) = A(t) + \lambda V(t)$.

□

Example 4.1. *The linearized mean curvature operator of a standard n-sphere $S_R(0)$ in $\mathbb{R}^{n+1}$ is $-\Delta_{S_R} - \frac{n}{R^2}$ and $\lambda_0 = 0$. Its kernel is spanned by $\{x^1/R, \ldots, x^n/R\}$, which corresponds translations in $\mathbb{R}^{n+1}$. Also note that because L_0 is self-adjoint, the cokernel equals the kernel.*

A classical result of Barbosa and do Carmo [1] characterizes stable hypersurfaces in Euclidean space.

Proposition 4.4 ([1])**.** *The only stable compact orientable immersed hypersurfaces in Euclidean space with non-zero constant mean curvature are round spheres.*

Proof. Let Σ be a stable compact orientable immersed CMC hypersurface. By Minkowski's integral formula,

$$\int_\Sigma (n - H\langle X, \nu\rangle) \, d\sigma_0 = 0$$

Consider the variation $\eta = n - H\langle X, \nu\rangle$. Then $\int_\Sigma \eta \, d\sigma_0 = 0$. Because Σ is stable and H is constant,

$$\int_\Sigma (-\eta \Delta_\Sigma \eta - |A|^2 \eta) \, d\sigma_0 \geq 0. \tag{4.3}$$

We claim that

$$\Delta_\Sigma \langle x, \nu\rangle = H - |A|^2 \langle x, \nu\rangle. \tag{4.4}$$

Let $p \in \Sigma$ and let $\{e_i\} \subset T\Sigma$ be a geodesic frame around p. Let ∇ be the connection of Euclidean space and ∇^Σ the connection of the induced metric. Then $\nabla^\Sigma_{e_i} e_j(p) = 0$ and $[e_i, e_j](p) = 0$. Because $\nabla_{e_i} X = e_i$ around p, we obtain, at p,

$$\begin{aligned}
\Delta_\Sigma \langle X, \nu\rangle &= e_i e_i \langle X, \nu\rangle = e_i \left\{\langle \nabla_{e_i} X, \nu\rangle + \langle X, \nabla_{e_i}\nu\rangle\right\} \\
&= \langle e_i, \nabla_{e_i}\nu\rangle + \langle X, \nabla_{e_i}\nabla_{e_i}\nu\rangle \\
&= H - |A|^2 \langle X, \nu\rangle,
\end{aligned}$$

where we use that $\langle e_k, \nabla_{e_i}\nabla_{e_i}\nu\rangle = 0$ if H is constant. Therefore,

$$\begin{aligned}
&-\eta\Delta_\Sigma \eta - |A|^2\eta \\
&= -(n - H\langle X,\nu\rangle)(-H^2 - H|A|^2\langle X,\nu\rangle) - |A|^2(n - 2nH\langle X,\nu\rangle + H^2\langle X,\nu\rangle^2) \\
&= nH^2 - nH|A|^2\langle X,\nu\rangle - H^3\langle X,\nu\rangle + H^2|A|^2\langle X,\nu\rangle^2 \\
&\quad - n^2|A|^2 + 2nH|A|^2\langle X,\nu\rangle - H^2|A|^2\langle X,\nu\rangle^2 \\
&= H^2\eta - n|A|^2\eta.
\end{aligned}$$

By (4.3) and the above inequality,

$$\begin{aligned}
0 &\le \int_\Sigma H^2\eta - n|A|^2\eta \, d\sigma_0 = -n\int_\Sigma |A|^2(n - H\langle X,\nu\rangle)\, d\sigma_0 \\
&= -n^2 \int_\Sigma |A|^2 \, d\sigma_0 + nH\int_\Sigma |A|^2\langle X,\nu\rangle\, d\sigma_0 \\
&= -\int_\Sigma (n|A|^2 - H^2)\, d\sigma_0 \qquad \text{(by applying Stokes theorem to (4.4))}.
\end{aligned}$$

Note a simple linear algebra fact says that $n|A|^2 - H^2 \ge 0$ with equality at $p \in \Sigma$ if and only if Σ is umbilic at p. Therefore, Σ is umbilic everywhere. Hence, Σ is a sphere. □

Remark. Note that the above result can be extended to simply connected complete ambient Riemannan manifold with constant sectional curvature [2].

5 Surfaces of constant mean curvature in asymptotically flat manifolds

Definition 5.1. *An asymptotically flat manifold (M, g) is called strongly asymptotically flat if*

$$g_{ij}(x) = \left(1 + \frac{2m}{|x|}\right)\delta_{ij} + O(|x|^{-2}).$$

We will discuss two proofs of the following theorem.

Theorem 5.1 ([10, 11]). *Let (M, g) be strongly asymptotically flat of $m \neq 0$. Given a constant $R \gg 1$, there exists a surface Σ of constant mean curvature $H = \frac{2}{R^2} - \frac{4m}{R^2} + O(R^{-3})$ and $||x| - R| \le C$ for any $x \in \Sigma$.*

Note that we shows that Σ is close to $S_R(\vec{C})$, the coordinate sphere center at the center of mass $\vec{C}$ [6]. Furthermore, we generalized the above result to a larger class of manifolds, which are asymptotically flat manifolds with the Regge-Teitelboim condition [7].

5.1 The volume-preserving mean curvature flow

In 1996, G. Huisken and S.-T. Yau initiated a program to study the unique foliation of constant mean curvature surfaces in an asymptotically flat manifold and use the foliation to define the geometric center of mass. They introduced the *volume-preserving mean curvature flow* in strongly asymptotically flat manifolds.

Let S_σ be the coordinate sphere centered at the origin in an asymptotically flat manifold (M, g). Define a family of maps $F^\sigma(\cdot, t) : S^2 \to (M, g)$ so that

$$\begin{aligned} &\frac{d}{dt} F^\sigma(x,t) = (\overline{H} - H)\nu(x,t) \quad t \geq 0, x \in S^2 \\ &F^\sigma(S^2, 0) = S_\sigma, \end{aligned} \tag{5.1}$$

where $\overline{H} = ⨏_{\Sigma_t} H \, d\sigma$ and $\Sigma_t = F^\sigma(S^2, t)$. By the variation formula of the enclosed volume (5.2), the flow keeps the volume enclosed by Σ_t constant. By (4.1), $A'(t) = -\int_{\Sigma_t} (H - \overline{H})^2 \, d\sigma$, so the $A(t)$ is decreasing. Hence, if the flow exists for infinite time, Σ_t is expected to converge to a solution that locally minimizes the area subject to the volume constraint.

Because the initial value problem (5.1) is a quasi-linear parabolic system, a unique short-time solution exists for smooth initial data. The long-time existence is rather delicate for the volume-preserving flow. A general initial solution to the volume-preserving flow may develop singularity. When (M, g) is Euclidean space, the long-time existence holds if Σ_0 is assumed uniformly convex. However, in other Riemannian manifolds (M, g), for example, in $\mathbb{S}^{n+1}$ such statement is not true [9, Remark on p. 38].

Theorem 5.2 ([9]). *Suppose that (M, g) is Euclidean space. If Σ_0 is uniformly convex, then the initial value problem* (5.1) *has a smooth solution for all $t > 0$ and Σ_t converge to a round sphere enclosing the same volume as Σ_0 in the C^∞-topology as $t \to \infty$.*

In the case that (M, g) is strongly asymptotically flat, if the initial surface is the coordinate spheres, then the following theorem holds.

Theorem 5.3 ([10]). *Let (M, g) be strongly asymptotically flat with $m > 0$. There is $\sigma_0(m) \gg 1$ so that for all $\sigma \geq \sigma_0$, the initial value problem* (5.1) *has a unique smooth solution for all $t > 0$. As $t \to \infty$, Σ_t converge exponentially fast to Σ^σ with constant mean curvature H_σ and*

$$|r(x) - \sigma| \leq C \text{ for all } x \in \Sigma^\sigma, \quad \text{and} \quad \left| H_\sigma - \frac{2}{\sigma} + \frac{4m}{\sigma^2} \right| \leq C\sigma^{-2}.$$

To prove the long time existence, we consider the class of round hypersurfaces $\mathfrak{B}_\sigma$. Let $r = |x|$. For $\sigma \geq 1$ and B non-negative real numbers, we define the class of surfaces in (M, g) by

$$\mathfrak{B}_\sigma = \{\Sigma \subset M : |r - \sigma| \leq B, |\mathring{A}| \leq B\sigma^{-3}, |\nabla^\Sigma \mathring{A}| \leq B\sigma^{-4}\}.$$

A classical result in differential geometry says that a compact hypersurface in Euclidean space with $|\mathring{A}| = 0$, i.e., umbilic, is a round sphere. The following proposition gives a quantitative generalization of the classical result:

Proposition 5.2. *Let (M^3, g) be asymptotically flat. Let $\Sigma^2 \subset (M, g)$. Denote by $r_1 = \frac{1}{10}\max_{x\in\Sigma}|x|$. Suppose that $|x| \geq r_1$ for all $x \in \Sigma$. Suppose that for some constants B,*

$$|\mathring{A}| \leq B r_1^{-3}, \quad \text{and} \quad |\nabla \mathring{A}| \leq B r_1^{-4}.$$

Then if $r_1 \gg 1$,

$$|\mathring{A}^e| \leq C r_1^{-3}, \quad \text{and} \quad |\nabla^e A^e| \leq C r_1^{-4}.$$

Moreover, there exist $r_0 \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$ so that

$$\begin{aligned} \left|\lambda_i - \frac{1}{r_0}\right| &\leq C r_1^{-3}, \\ |(y - \vec{a}) - r_0 \nu_e| &\leq C r_1^{-1}, \\ \left|\nu_e - \frac{y - \vec{a}}{r_0}\right| &\leq C r_1^{-2}. \end{aligned}$$

By the above proposition, if $\sigma \gg B$, then $\Sigma \in \mathfrak{B}_\sigma$ is sufficiently round. In particular, let H be the mean curvature of $|\Sigma$, then

$$H = \frac{2}{r_0} - \frac{4m}{r_0^2} + O(r_0^{-3}).$$

Theorem 5.4. *If $\sigma \gg 1$, the solution to* (5.1) *remains in $\mathfrak{B}_\sigma$.*

Theorem 5.5. *The mean curvature H of Σ_t converges exponentially fast to a constant.*

Proof. By Proposition 4.1,

$$\begin{aligned} \frac{d}{dt} d\sigma &= H(\overline{H} - H)\, d\sigma \\ \frac{d}{dt} H &= \Delta_{\Sigma_t} H - (\overline{H} - H)(|A|^2 + Ric(\nu, \nu)) \\ \frac{d}{dt}\int_{\Sigma_t} d\sigma &= -\int_{\Sigma_t} (H - \overline{H})^2\, d\sigma. \end{aligned} \tag{5.2}$$

Because $\int_{\Sigma_t}(H - \overline{H})\, d\sigma = 0$ for all t,

$$\begin{aligned} &\frac{d}{dt}\int_{\Sigma_t} (H - \overline{H})^2\, d\sigma = \int_{\Sigma_t} 2(H - \overline{H})\frac{d}{dt}(H - \overline{H})\, d\sigma - \int_{\Sigma_t} (H - \overline{H})^3 H\, d\sigma \\ &= \int_{\Sigma_t} (H - \overline{H})(\Delta_{\Sigma_t} H - (\overline{H} - H)(|A|^2 + Ric(\nu, \nu)))\, d\sigma - \int_{\Sigma_t} (H - \overline{H})^3 H\, d\sigma \\ &= -2\int_{\Sigma_t} (H - \overline{H}) L_{\Sigma_t}(H - \overline{H})\, d\sigma - \int_{\Sigma_t} (H - \overline{H})^3 H\, d\sigma \\ &= \left(-\frac{12m}{\sigma^3} + O(\sigma^{-4})\right)\int_{\Sigma_t} (H - \overline{H})^2\, d\sigma - \int_{\Sigma_t} (H - \overline{H})^3 H\, d\sigma \qquad \text{(by Lemma 7.1.)} \end{aligned}$$

Note that by (5.2),

$$\int_0^\infty \int_{\Sigma_t} (H - \overline{H})^2 \, d\sigma = |\Sigma_t| - |\Sigma_\infty| \le |\Sigma_t|.$$

Therefore, $\max_{\Sigma_t} |H - \overline{H}|$ tends to zero as t tends to infinity. Therefore, for some $t_0 \gg 1$, $\max_{\Sigma_t} |H(H - \overline{H})| \le \epsilon m \sigma^{-3}$. Hence, for $\sigma \gg 1$,

$$\frac{d}{dt} \int_{\Sigma_t} (H - \overline{H})^2 \, d\sigma \le -\frac{6m}{\sigma^3} \int_{\Sigma_t} (H - \overline{H})^2 \, d\sigma$$

□

5.2 The perturbation method

Another method to construct surfaces with constant mean curvature in the asymptotically flat manifold is by perturbation. The coordinate sphere in (M, g) has constant mean curvature with respect to the Euclidean metric. Hence the mean curvature of a large coordinate sphere in an asymptotically flat manifold should be *close* to a constant, and one expect to find a hypersurface with constant mean curvature nearby.

Let (M, g) be strongly asymptotically flat so that

$$g_{ij} = \left(1 + \frac{2m}{|y|}\right) \delta_{ij} + p_{ij}.$$

Let $S_R(p)$ be a coordinate sphere centered at p of radius R. Denote $\rho^i = \frac{y^i - p^i}{R}$. Then the mean curvature of $S_R(a)$ at $y \in S_R(a)$ is

$$\begin{aligned} H_{S_R(a)} &= \frac{2}{R} - \frac{4m}{R^2} + \frac{6m(y-a)\cdot a}{R^4} + \frac{9m^2}{R^3} \\ &\quad + \frac{1}{2} p_{ij,k}(y) \rho^i \rho^j \rho^k + 2 \frac{p_{ij}(y)}{R} \rho^i \rho^j \\ &\quad - p_{ij,i}(y) \rho^j - \frac{p_{ii}(y)}{R} + \frac{1}{2} p_{ii,j}(y) \rho^j + O(R^{-4}(1 + |a|)) \\ &= \frac{2}{R} - \frac{4m}{R^2} + \frac{6m(y-a)\cdot a}{R^4} + \frac{9m^2}{R^3} + G(y, a, R) \end{aligned} \tag{5.3}$$

Theorem 5.6 ([11, 6])**.** *Let (M, g) be strongly asymptotically flat. Given $R \gg 1$, there exists a surface Σ of constant mean curvature $\frac{2}{R} - \frac{4m}{R^2}$. Moreover,*

$$\Sigma = \{x + \phi\nu : x \in S_R(\vec{C}), \phi \in C^{2,\alpha}(S_R(p))\},$$

where $\vec{C}$ is the center of mass and $\|\phi\|_{C^{,\alpha}} \le CR^{-1}$.

Proof. Let ν be the outward unit normal to $S_R(a)$. We consider the perturbation of $S_R(a)$

$$\Sigma = \{x + \phi\nu : x \in S_R(a), \phi \in C^{2,\alpha}(S_R(p))\}.$$

Denote by $\mathcal{H}_R(a,\phi) : \mathbb{R}^n \times C^{2,\alpha}(S_R(a)) \to C^{0,\alpha}(S_R(a))$ the mean curvature of Σ in (M,g). By Taylor expansion in the ϕ-component for mappings between two Banach Spaces,

$$H(a,R,\phi) = H(a,R,0) + dH(a,R,0)\phi + \int_0^1 \left(dH(a,R,s\phi) - dH(a,R,0)\right)\phi\, ds,$$

where where dH and d^2H are the first and second Fréchet derivatives, and $dH(a,R,s\phi)$ is the linearized mean curvature operator. Fix $R \gg a$. By (5.3), to solve $H(a,R,\phi) = \frac{2}{R} - \frac{4m}{R^2}$ for some $(a,\phi) \in \mathbb{R}^n \times C^{2,\alpha}(S_R(a))$ is equivalent to solving

$$L_{S_R(a)}\phi = -\frac{6m(y-a)\cdot a}{R^4} - \frac{9m^2}{R^3} - G(y,a,R) - \int_0^1 \left(dH(s\phi) - dH(0)\right)\phi\, ds. \tag{5.4}$$

Note that $|A|^2 = \frac{2}{R^2} + O(R^{-3})$ and $Ric(\nu,\nu) = O(|y|^{-3}) = O(R^{-3})$ if $|a| \le R$. Denote by $L_0 := -\overline{\Delta} - \frac{2}{R^2}$, where $\overline{\Delta}$ is the Laplacian for the standard Euclidean sphere $S_R(p)$. Then we can rewrite (5.4) as

$$\begin{aligned} L_0\phi &= -\frac{6m(y-a)\cdot a}{R^4} - \frac{9m^2}{R^3} - G(y,a,R) + O(R^{-1}|D^2\phi| + R^{-2}|D\phi| + R^{-3}) \\ &=: F(a,R,\phi,D\phi,D^2\phi). \end{aligned}$$

A necessary condition to solve the above equation is that F is perpendicular to the cokernel of L_0 spanned by $\{y^1-a^1, y^2-a^2, y^3-a^3\}$. In order to do this, we choose the correct center p. By Lemma 6.1 in the next section,

$$\int_{S_R(a)} F(a,R,\phi,D\phi,D^2\phi)\frac{y^i-a^i}{R}\, d\sigma_0 = m(a^i - \vec{C}^i) + O(R^{-1}\|\phi\|_{C^2}).$$

Hence, we choose $a = \vec{C} + O(R^{-1}\phi)$ so that the right hand side is zero. Therefore, given any ϕ, there exists a unique a so that $F(a,R,\phi,D\phi,D^2\phi)$ is in the range of L_0.

Denote by $\mathcal{B} := \{u \in C^2(S_R(p)) : \|u\|_{C^{2,\alpha}} = 1\}$. $\mathcal{B}$ is compact and convex. Define $T : \mathcal{B} \to C^2(S_R(p))$ by $T(w) = v$, where v is the unique solution in $\mathrm{Ker}^\perp$ so that $Lv = F(a,R,w,Dw,D^2w)$. By the Schauder estimate and the fact that $v \in \mathrm{Ker}^\perp$,

$$\|v\|_{C^{2,\alpha}} \le C\|F(a,R,w,Dw,D^2w)\|_{C^{0,\alpha}} \le CR^{-1}\|w\|_{C^{2,\alpha}},$$

where $C = C(C_0,m)$. For $R \ge C$, T maps the $\mathcal{B}_1$ to itself. It is standard to check that T is continuous. Therefore, by the Schauder fixed point theorem (Theorem A.1), T has a fixed point ϕ, which solves $L_0\phi = F(a,R,\phi,D\phi,D^2\phi)$.

□

6 Equivalence of center of mass

Lemma 6.1 ([6])**.** *Let (M,g) be asymptotically flat satisfying the Regge-Teitelboim condition. Then for $\alpha = 1,2,3$,*

$$\int_{S_R(p)} (x^\alpha - p^\alpha)\left(H_S - \frac{2}{R}\right) d\sigma_0 = 8\pi m(p^\alpha - C^\alpha) + O(R^{1-2q}). \tag{6.1}$$

Proof. Denote by $h_{ij} = g_{ij} - \delta_{ij}$ and denote $\rho^i = \frac{x^i - p^i}{R}$. By direct computation,

$$\begin{aligned} H_S(x) =& \frac{2}{R} + \frac{1}{2}\sum_{i,j,k} h_{ij,k}(x)\rho^i\rho^j\rho^k + 2\sum_{i,j} h_{ij}(x)\frac{\rho^i\rho^j}{R} - \sum_{i,j} h_{ij,i}(x)\rho^j \\ &+ \frac{1}{2}\sum_{i,j} h_{ii,j}(x)\rho^j - \sum_i \frac{h_{ii}(x)}{R} + E_0(x), \end{aligned}$$

where $E_0(x) = O(R^{-1-2q})$ and $E_0^{odd}(x) = O(R^{-2-2q})$.

We claim that

$$\begin{aligned} &\int_{S_R(p)} (x^\alpha - p^\alpha)\frac{1}{2}\sum_{i,j,k} h_{ij,k}(x)\rho^i\rho^j\rho^k \, d\sigma_0 \\ &= \int_{S_R(p)} (x^\alpha - p^\alpha)\sum_{i,j}\left[\frac{1}{2}h_{ij,i}(x)\rho^j - 2h_{ij}(x)\frac{\rho^i\rho^j}{R}\right] d\sigma_0 \\ &+ \int_{S_R(p)} \frac{1}{2}\sum_i \left[h_{ii}(x)\rho^\alpha + h_{i\alpha}(x)\rho^i\right] d\sigma_0. \end{aligned}$$

The original proof used a density theorem. Recently, a simple proof is provided by [5]. To see this, let $X_{(\alpha)} = \sum_{i,j}(x^\alpha - p^\alpha)h_{ij}\rho^i\partial_j$. Note that

$$\int_{S_R(p)} \mathrm{div}_0 X \, d\sigma_0 = \int_{S_R(p)} H_0\delta(X,\rho)\, d\sigma_0$$

and

$$\begin{aligned} \mathrm{div}_0 X &= \left(\delta_{ij} - \rho^i\rho^j\right)\partial_i X^j \\ &= h_{il}\rho^i + (x^l - p^l)\left(\frac{h_{ii}}{R} - 2\frac{h_{ij}}{r}\rho^i\rho^j + h_{ij,j}\rho^i - h_{ij,k}\rho^i\rho^j\rho^k\right). \end{aligned}$$

We then have

$$\begin{aligned} &\int_{S_R(p)} (x^\alpha - p^\alpha)\left(H_S - \frac{2}{R}\right) d\sigma_0 \\ &= -\frac{1}{2}\int_{S_R(p)} (x^\alpha - p^\alpha)\sum_{i,j}(h_{ij,i} - h_{ii,j})\rho^j \, d\sigma_0 + \frac{1}{2}\int_{S_R(p)} \sum_i (h_{i\alpha}\rho^i - h_{ii}\rho^\alpha)\, d\sigma_0 + O(R^{1-2q}) \\ &= -\frac{1}{2}\int_{S_R(p)} x^\alpha \sum_{i,j}(g_{ij,i} - g_{ii,j})\rho^j - \sum_i (g_{i\alpha}\rho^i - g_{ii}\rho^\alpha)\, d\sigma_0 \\ &\quad + \frac{1}{2}p^\alpha \int_{S_R(p)} \sum_{i,j}(g_{ij,i} - g_{ii,j})\rho^j \, d\sigma_0 + O(R^{1-2q}) \\ &= -\frac{1}{2}\int_{S_R(p)} x^\alpha \sum_{i,j}(g_{ij,i} - g_{ii,j})\frac{x^j}{|x|} - \sum_i (g_{i\alpha}\frac{x^i}{|x|} - g_{ii}\frac{x^\alpha}{|x|})\, d\sigma_0 \\ &\quad + \frac{1}{2}p^\alpha \int_{S_R(p)} \sum_{i,j}(g_{ij,i} - g_{ii,j})\frac{x^j}{|x|}\, d\sigma_0 - \frac{1}{2R}\int_{S_R(p)} \sum_j Z^j_{(\alpha)}p^j \, d\sigma + O(R^{1-2q}), \end{aligned}$$

where $Z^j_{(\alpha)} = \sum_{i,j} x^\alpha(g_{ij,i} - g_{ii,j}) - \sum_i (g_{i\alpha} - g_{ii}\delta_{i\alpha})\partial_j$. By the Regge-Teitelboim condition, $Z^j_{(\alpha)}(x) - Z^j_{(\alpha)}(-x) = O(R^{-1-q})$, so the last two terms above are of order $O(R^{1-2q})$. Then using the definitions of the ADM mass and center of mass, we prove the lemma. □

7 Foliations of stable constant mean curvature surfaces

We shall show that the constant mean curvature surface that we constructed in the previous section is stable.

Lemma 7.1. *Let (M, g) be strongly asymptotically flat with $m > 0$ and let Σ be the surface of constant mean curvature constructed in the previous section. Then Σ is stable and the lowest eigenvalues among test functions with zero mean is*

$$\lambda_0 = \frac{6m}{R^3} + O(R^{-2-2q}).$$

Proof. It follows by computing each term in the stability operator directly. First, let μ_{Lap} be the lowest eigenvalue of $-\Delta_\Sigma$ with respect to test functions which have zero mean. Recall the Bochner–Lichnerowicz identity:

$$\begin{aligned}\frac{1}{2}\Delta_\Sigma|\nabla^\Sigma u|^2 &= (\mathrm{Hess}_\Sigma u)^2 + \langle \nabla^\Sigma u, \nabla^\Sigma \Delta_\Sigma u\rangle + \mathcal{K}(\nabla^\Sigma u, \nabla^\Sigma u)\\ &\geq \frac{(\Delta_\Sigma u)^2}{2} + \langle \nabla^\Sigma u, \nabla^\Sigma \Delta_\Sigma u\rangle + \mathcal{K}(\nabla^\Sigma u, \nabla^\Sigma u),\end{aligned}$$

where $\mathcal{K}$ is the Gauss curvature[1] of Σ. By letting u be the eigenfunction $-\Delta_\Sigma u = \mu_{Lap} u$, we can show that $\mu_{Lap} \geq 2\mathcal{K}$ and then

$$\mu_{Lap} \geq \frac{2}{R^2} - \frac{4m}{R^3} + O(R^{-4}). \tag{7.1}$$

On the other hand,

$$|A|^2 + Ric(\nu,\nu) = \frac{2}{R^2} - \frac{10m}{R^3} + O(R^{-4}). \tag{7.2}$$

We complete the proof. □

Lemma 7.2. *The stability operator L_Σ is invertible if $R \gg 1$.*

Remark. The operator may not be invertible if R is small. For example, for the round spheres in the Schwarzschild solution, $\mu_0 = 0$ for $L_{S_{\frac{(2+\sqrt{3})m}{2}}}$.

[1] Ricci curvature in higher dimensions.

Proof. We shall analyze the eigenvalues of L_Σ. Let μ_0 be the lowest eigenvalue of L_Σ and μ_1 be the next eigenvalue. We will prove that $\mu_0 < 0$ and $\mu_1 > 0$ if $R \gg 1$.

Let w be an eigenfunction with respect to μ_0

$$-\Delta_\Sigma w - (|A|^2 + Ric(\nu,\nu))w = \mu_0 w. \tag{7.3}$$

We first claim that w is close to a constant function in L^2. By (7.2),

$$\begin{aligned}\mu_0 \int_\Sigma w^2\, d\sigma &= \int_\Sigma \left(|\nabla w|^2 - (|A|^2 + Ric(\nu,\nu))w^2\right)\, d\sigma \\ &\geq \left(-\frac{2}{R^2} + \frac{10m}{R^3} + O(R^{-4})\right)\int_\Sigma w^2\, d\sigma.\end{aligned}$$

On the other hand, choosing constant functions as comparison functions, we have

$$\mu_0 \leq -\frac{2}{R^2} + \frac{10m}{R^3} + O(R^{-4}). \tag{7.4}$$

Therefore,

$$\mu_0 = -\frac{2}{R^2} + \frac{10m}{R^3} + O(R^{-4}).$$

Multiply (7.3) by $w - \bar{w}$ and integrate the identity over Σ using the divergence theorem. We then have

$$\begin{aligned}\int_\Sigma |\nabla(w-\bar{w})|^2\, d\sigma &= \int_\Sigma (\mu_0 + |A|^2 + Ric(\nu,\nu))(w-\bar{w})^2\, d\sigma \\ &\quad + \int_\Sigma (|A|^2 + Ric(\nu,\nu))\bar{w}(w-\bar{w})\, d\sigma.\end{aligned}$$

By (7.1), (7.2), and (7.4), we obtain for $R \gg 1$,

$$\left(\frac{2}{R^2} + O(R^{-3})\right)\int_\Sigma |w-\bar{w}|^2\, d\sigma \leq \int_\Sigma O(R^{-4})\left(|w-\bar{w}|^2 + |\bar{w}||w-\bar{w}|\right)\, d\sigma.$$

Apply Cauchy-Schwarz inequality to the last integrand

$$|\bar{w}||w-\bar{w}| \leq \epsilon R^2 |w-\bar{w}|^2 + C(\epsilon)R^{-2}|\bar{w}|^2.$$

Therefore, for $R \gg 1$,

$$\int_\Sigma |w-\bar{w}|^2\, d\sigma \leq CR^{-4}\int_\Sigma \bar{w}^2\, d\sigma. \tag{7.5}$$

Let u be the eigenfunction with respect to μ_1. We shall show that $\mu_1 \geq \frac{6m}{R^3} + O(R^{-4})$. Because $L_\Sigma u = \mu_1 u$,

$$L_\Sigma(u-\bar{u}) = \mu_1(u-\bar{u}) + (\mu_1 + |A|^2 + Ric(\nu,\nu))\bar{u}.$$

Multiply the above identity by $(u-\bar{u})$ and integrate it over Σ. By Lemma 7.1, we obtain

$$\begin{aligned}
&\left(\frac{6m}{R^3}+O(R^{-4})\right)\int_\Sigma |u-\bar{u}|^2\,d\sigma \\
&\le \mu_1\int_\Sigma (u-\bar{u})^2\,d\sigma+\int_\Sigma(\mu_1+|A|^2+Ric(\nu,\nu))\bar{u}(u-\bar{u})\,d\sigma \\
&= \mu_1\int_\Sigma (u-\bar{u})^2\,d\sigma+\int_\Sigma O(R^{-4})\bar{u}(u-\bar{u})\,d\sigma \\
&\le \mu_1\int_\Sigma (u-\bar{u})^2\,d\sigma+O(R^{-4})|\bar{u}||\Sigma|^{1/2}\|u-\bar{u}\|_{L^2} \qquad \text{(by Hölder inequality).}
\end{aligned} \tag{7.6}$$

To estimate $\bar{u}$, we note that

$$0=\int_\Sigma uw\,d\sigma=\int_\Sigma (u-\bar{u})(w-\bar{w})\,d\sigma+\int_\Sigma u\bar{w}\,d\sigma.$$

Hence, by Hölder inequality and (7.5) (we can assume $\bar{u}\ge 0$),

$$|\bar{u}||\Sigma|=\bar{u}|\Sigma|=\int_\Sigma u\,d\sigma\le|\bar{w}|^{-1}\|u-\bar{u}\|_{L^2}\|w-\bar{w}\|_{L^2}\le CR^{-2}|\Sigma|^{1/2}\|u-\bar{u}\|_{L^2}.$$

Plugging the above inequality into (7.6), we have

$$\left(\frac{6m}{R^3}+O(R^{-4})\right)\|u-\bar{u}\|^2_{L^2}\le\mu_1\|u-\bar{u}\|^2_{L^2}+O(R^{-6})\|u-\bar{u}\|^2_{L^2}.$$

Hence,

$$\mu_1=\frac{6m}{R^3}+O(R^{-4}).$$

□

Remark. By the inverse function theorem, the surface of constant mean curvature is unique in a neighborhood of Σ.

Using similar analysis, one can show that if $L_\Sigma u\approx$ constant. Then u is approximately a constant, which in particular has a sign. Hence, the CMC surfaces do not intersect.

Theorem 7.1. *The surfaces of constant mean curvature constructed above form a smooth foliation.*

8 Density theorems

Among the space of asymptotically flat data, it may be sometimes essential to find a "better" class of representatives. Here, we consider the space of asymptotically flat data with harmonic asymptotics, i.e.,

$$g_{ij}=u^4\delta_{ij}\qquad\text{and}\qquad\pi_{ij}=u^2(\mathcal{L}_\delta X)_{ij},$$

where $\pi=k-(\mathrm{tr}_g k)g$ the operator $\mathcal{L}_g$ is the operator related to the Lie derivative $L_X g$ by $\mathcal{L}_g X=L_X g-\mathrm{div}_g(X)g$.

Definition 8.1 (Weighted Sobolev Spaces)**.** *For a non-negative integer k, a non-negative real number p, and a real number δ, we say $f \in W^{k,p}_{-\delta}(M)$ if*

$$\|f\|_{W^{k,p}_{-\delta}(M)} \equiv \left(\int_M \sum_{|\alpha| \le k} \left(\left| D^\alpha f \right| \rho^{|\alpha|+\delta} \right)^p \rho^{-3} \, d\sigma_g \right)^{\frac{1}{p}} < \infty,$$

where α is a multi-index and ρ is a continuous function with $\rho = |x|$ on $M \setminus B_R$. When $p = \infty$,

$$\|f\|_{W^{k,\infty}_{-\delta}(M)} = \sum_{|\alpha| \le k} ess \sup_M |D^\alpha f| \rho^{|\alpha|+\delta}.$$

Proposition 8.2 ([4])**.** *Let $p > n$, $q \in (\frac{n-2}{2}, n-2)$, $q_0 > 1$. Suppose that (M^n, g, π) is an asymptotically flat initial data set*

$$(g_{ij} - \delta_{ij}, \pi_{ij}) \in W^{2,p}_{-q}(M) \times W^{1,p}_{-1-q}(M)$$

and

$$(\mu, J) \in C^{0,\alpha}_{-n-q_0}.$$

such that

$$g = u^s \delta, \quad \pi = u^{s/2} \mathcal{L}_\delta X, \tag{8.1}$$

outside a large ball B for some $(u-1, X) \in W^{2,p}_{-q}$. Then (g, π) has harmonic asymptotics

$$u(x) = 1 + a|x|^{2-n} + O^{2,\alpha}(|x|^{1-n}) \qquad \text{and} \qquad X_i(x) = b_i |x|^{2-n} + O^{2,\alpha}(|x|^{1-n}).$$

Theorem 8.1 ([3])**.** *Let (M, g, π) be a vacuum asymptotically flat initial data set. For every $\epsilon > 0$ there exists asymptotically flat initial data $(\bar{g}, \bar{\pi})$ with harmonic asymptotics, such that we have*

$$\|g - \bar{g}\|_{W^{2,p}_{-q}} < \epsilon \quad \text{and} \quad \|\pi - \bar{\pi}\|_{W^{1,p}_{-1-q}} < \epsilon, \tag{8.2}$$

and

$$|E - \bar{E}| < \epsilon \quad \text{and} \quad |P - \bar{P}| < \epsilon. \tag{8.3}$$

It suffices to prove (8.2) because by the following proposition, (8.2) implies (8.3).

Proposition 8.3. *Let $p > n$, $q \in (\frac{n-2}{2}, n-2)$. Let (g, π) and $(\bar{g}, \bar{\pi})$ be asymptotically flat initial data sets so that*

$$(g_{ij} - \delta_{ij}, k_{ij}) \in W^{2,p}_{-q}(\mathbb{R}^n \setminus B) \times W^{1,p}_{-1-q}(\mathbb{R}^n \setminus B)$$

and

$$(\mu, J) \in C^{0,\alpha}_{-n-q_0}.$$

Given $\epsilon > 0$ there exists $\delta > 0$ (depending only on $\epsilon, n, p, q, q_0, \|(g-\delta, \pi)\|_{W^{2,p}_{-q}\times W^{1,p}_{-q-1}}$, and $\|(\mu, J) - (\bar{\mu}, \bar{J})\|_{W^{0,1}_{-n-q_0/2}}$) such that if

$$\|g - \bar{g}\|_{W^{2,p}_{-q}} \leq \delta \quad \textit{and} \quad \|\pi - \bar{\pi}\|_{W^{1,p}_{-1-q}} \leq \delta,$$

then

$$|E - \bar{E}| < \epsilon \quad \textit{and} \quad |P - \bar{P}| < \epsilon.$$

Proof of Theorem 8.1. Consider the cut-off data set. For $\lambda \geq 1$ large define the cut-off initial data

$$(\hat{g}_\lambda)_{ij} = \chi_\lambda g_{ij} + (1-\chi_\lambda)\delta_{ij}, \qquad \hat{\pi}_\lambda = \chi_\lambda \pi,$$

where $\chi_\lambda(x) = \chi(x/\lambda)$ and χ is a $\mathcal{C}^{3,\alpha}$ cut-off function on $\mathbb{R}^n$ that is 1 on $\{|x| \leq 1\}$ and 0 on $\{|x| \geq 2\}$. Note that $\|(\hat{g}_\lambda - g, \hat{\pi}_\lambda - \pi)\|_{W^{2,p}_{-q}\times W^{1,p}_{-1-q}} \to 0$ as $\lambda \to \infty$. However, the cut-off data set is not vacuum in the annulus.

The system of constraint equations is under-determined, i.e., there are fewer equations than unknowns. To make it well-determined, we consider the unknowns $u, X_i, i = 1, 2.3$ and make the ansatz of asymptotics the data set

$$\tilde{g} = u^4 \hat{g} \qquad \text{and} \qquad \tilde{\pi} = u^2(\hat{\pi} + \mathcal{L}_{\hat{g}} X).$$

We want to find u and X so that $(\tilde{g}, \tilde{\pi})$ satisfies the vacuum constraints. Let T be the constraint map $T(u, X) = \Phi(\tilde{g}, \tilde{\pi})$ whose first and last three components are

$$\begin{aligned} T(u,X) = &\Big(8u^{-1}\Delta_{\hat{g}} u - R_{\hat{g}} - \tfrac{1}{2}\left(\mathrm{tr}_{\hat{g}}\pi + \mathrm{tr}_{\hat{g}}\mathcal{L}_{\hat{g}}X\right)^2 + \left(|\pi|^2_{\hat{g}} + 2(\mathcal{L}_{\hat{g}}X)_{kl}\pi^{kl} + |\mathcal{L}_{\hat{g}}X|^2_{\hat{g}}\right), \\ &(\mathrm{div}_{\hat{g}}\mathcal{L}_{\hat{g}}X + \mathrm{div}_{\hat{g}}\pi)_j + 4u^{-1}u_{,k}(\pi + \mathcal{L}_{\hat{g}}X)^k_j - 2u^{-1}u_{,j}\mathrm{tr}_{\hat{g}}(\pi + \mathcal{L}_{\hat{g}}X)\Big). \end{aligned} \tag{8.4}$$

Here, all indices are raised with respect to $\hat{g}$. The linearization of $T_{(\hat{g},\hat{\pi})}$ at $(1,0)$ is

$$\begin{aligned} &DT_{(\hat{g},\hat{\pi})}|_{(1,0)}(v, Z) \\ &= \Big(8\Delta_{\hat{g}}v + 4\big[R_{\hat{g}} - |\hat{\pi}|^2_{\hat{g}} + \tfrac{1}{2}(\mathrm{tr}_{\hat{g}}\hat{\pi})^2\big]v - 4Z_{k;l}\hat{\pi}^{kl} - \tfrac{2}{n-1}\mathrm{tr}_{\hat{g}}\hat{\pi}\,\mathrm{div}_{\hat{g}}Z, \\ &\quad \mathrm{div}_{\hat{g}}(\mathcal{L}_{\hat{g}}Z)_j + 4v_{,k}\hat{\pi}^k_j - 2v_{,j}\mathrm{tr}_{\hat{g}}\hat{\pi} - 2(\mathrm{div}_{\hat{g}}\hat{\pi})_j v\Big), \end{aligned}$$

where indices are raised and covariant derivatives are taken with respect to $\hat{g}$. The map $T_{(g,\pi)} : (W^{2,p}_{-q} + 1) \times W^{2,p}_{-q} \to W^{0,p}_{-2-q}$ is defined analogously. Because $q \in (\frac{1}{2}, 1)$ and $p > 3$, $DT_{(\hat{g},\hat{\pi})}|_{(1,0)}$ and $DT_{(g,\pi)}|_{(1,0)}$ are Fredholm operators of index 0 for λ sufficiently large.

Let K_1 be a complementing subspace for the kernel of $DT_{(g,\pi)}|_{(1,0)}$ in $W^{2,p}_{-q} \times W^{1,p}_{-2-q}$. Since the linearization $D\Phi|_{(g,\pi)} : W^{2,p}_{-q} \times W^{1,p}_{-1-q} \to W^{0,p}_{-2-q}$ is surjective and because $DT_{(g,\pi)}|_{(1,0)}$ is Fredholm we can find $C^{3,\alpha}$ compactly supported symmetric $(0,2)$-tensors $\{(h_k, w_k)\}_{k=1}^N$ whose images $D\Phi|_{(g,\pi)}(h_k, w_k)$ form a basis for a complementing subspace of the image of $DT_{(g,\pi)}|_{(1,0)}$

in $W^{0,p}_{-2-q}$. Let $K_2 = \mathrm{span}\{(h_k, w_k)\}_{k=1}^N$. We define the maps $\overline{T}_{(\hat{g},\hat{\pi})}, \overline{T}_{(g,\pi)} : K_1 \times K_2 \to W^{0,p}_{-2-q}$ by

$$\overline{T}_{(\hat{g},\hat{\pi})}(u, X, h, w) = \Phi(u^s \hat{g} + h, u^{s/2}(\hat{\pi} + \mathcal{L}_{\hat{g}} X) + w),$$

and

$$\overline{T}_{(g,\pi)}(u, X, h, w) = \Phi(u^s g + h, u^{s/2}(\pi + \mathcal{L}_g X) + w).$$

Note that the maps $\overline{T}_{(\hat{g},\hat{\pi})}, \overline{T}_{(g,\pi)}$ are continuously differentiable. Using that $(\hat{g}, \hat{\pi})$ converges to (g, π) in $W^{2,p}_{-q} \times W^{1,p}_{-q-1}$ as $\lambda \to \infty$ it is easy to see that $D\overline{T}_{(\hat{g},\hat{\pi})}|_{(u,X,h,w)}$ converges to $D\overline{T}_{(g,\pi)}|_{(u,X,h,w)}$ as $\lambda \to \infty$ locally uniformly in $(u, X, h, w) \in K_1 \times K_2$ in the strong operator topology. Observe that $D\overline{T}_{(g,\pi)}|_{(1,0,0,0)}$ is an isomorphism by construction. We conclude from the inverse function theorem that there exists $\delta_0 > 0$ such that for all $\lambda \geq 1$ sufficiently large, $\overline{T}_{(g,\pi)}$ restricts to a C^1 diffeomorphism defined on an open neighborhood of $(1, 0, 0, 0)$ (independent of $\lambda \geq 1$) in $K_1 \times K_2$ and onto an open neighborhood containing the $W^{0,p}_{-2-q}$-ball of radius $2\delta_0$ centered at $(0, 0)$. □

The above construction has some applications. For example, one can specify the angular momentum and center of mass.

Theorem 8.2 ([8])**.** *Let (g, π) be a nontrivial vacuum initial data set satisfying the Regge–Teitelboim condition. Given any constant vectors $\vec{\alpha}_0, \vec{\gamma}_0 \in \mathbb{R}^3$, there exists a vacuum initial data set $(\bar{g}, \bar{\pi})$ within a small neighborhood of (g, π) in $W^{2,p}_{-q} \times W^{1,p}_{-1-q}$ and*

$$\overline{E} = E, \quad \overline{P} = P,$$

and

$$\overline{J} = J + \vec{\alpha}_0, \quad \overline{C} = C + \vec{\gamma}_0.$$

The construction of $(\bar{g}, \bar{\pi})$ in the above theorem is similar to that of Theorem 8.1. Instead using the cut-off data, we consider

$$\hat{g} = g + \sigma \qquad \text{and} \qquad \hat{\pi} = \pi + \tau$$

where the $(0, 2)$ tensors $\sigma, \tau \in C^\infty_0(A_k)$ satisfying the linearized constraint equaitions

$$\begin{aligned} L\sigma &= 0 \\ \sum_i \tau_{ij,i} &= 0, \quad \text{for } j = 1, 2, 3. \end{aligned} \tag{8.5}$$

To specify the angular momentum, we require that σ and τ satisfy the additional condition $\sigma(x) = \sigma(-x), \tau(x) = \tau(-x)$ and

$$\int_{A_1} \left[\frac{1}{2} \tau_{ij,l} (Y_k)^l + \tau_{il} (Y_k)^l_{,j} \right] \sigma^{ij} \, dx = -8\pi\alpha_k \tag{8.6}$$

for $k = 1, 2, 3$ where $Y_k = \frac{\partial}{\partial x^k} \times \vec{x}$ (cross product). To specify the center of mass, we require σ to be a trace-free and divergence-free symmetric $(0, 2)$ tensor $\sigma \in C^\infty_0(A_1)$ so that

$$\int_{A_1} x^p \sum_{i,j,k} (\sigma_{ij,k})^2 \, dx = 64\pi E \gamma_p \tag{8.7}$$

for $p = 1, 2, 3$.

Remark. It is known that the mass-angular momentum inequality $m \geq \sqrt{|J|}$ holds for axial-symmetric AF data. Theorem 8.2 says that such inequality does not hold in general asymptotically flat data with no axial-symmetry.

A Basic facts

Theorem A.1 (Schauder Fixed Point Theorem)**.** *Let $\mathcal{B}$ be a compact convex subset in a Banach space and let $T : \mathcal{B} \to \mathcal{B}$ be continuous. Then T has a fixed point, that is, $Tx = x$ for some $x \in \mathcal{B}$.*

Theorem A.2 (Inverse Mapping Theorem)**.** *Let E and F be Banach spaces and let U be an open subset of E. Let $f : U \subset E \to F$ be of class C^r, $r \geq 1$, $x_0 \in U$. Suppose that $Df(x_0)$ is a linear isomorphism. Then f is a C^r diffeomorphism of some neighborhood of x_0 onto some neighborhood of $f(x_0)$.*

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