Summer 2012 MSRI SGW on Mathematical General Relativity
Problems on Euclidean harmonic functions

1. a. Verify that the following distributional equations hold: $\Delta(\frac{1}{2\pi}\log|x|) = \delta_0$ in dimension $n = 2$, while $\Delta(\frac{1}{(2-n)n\omega_n}|x|^{2-n}) = \delta_0$ in dimensions $n > 2$. Here δ_0 is the Dirac delta distribution at the origin.

b. Suppose $f \in C^2_c(\mathbb{R}^n)$, $n > 2$. Suppose $\mathrm{spt}(f) \subset \{x : |x| \leq K\}$. Then if we let $u(x) = \frac{1}{(2-n)n\omega_n}\int\limits_{\mathbb{R}^n} |x-y|^{2-n} f(y)\, dy$, then $\Delta u = f$ by the above. Moreover, show that u has an expansion of the form $u(x) = \frac{A}{|x|^{n-2}} + \frac{B_i x^i}{|x|^n} + O(|x|^{-n})$. Express the constants A and B_i in terms of integrals involving f.

2. a. Using the idea of the proof of the Removable Singularities Theorem, show that if u is harmonic with an isolated singularity at $x = 0$, show that the singularity is in fact removable if $\lim\limits_{x\to 0}|x|^{n-2}u(x) = 0$ in case $n > 2$, and in case $n = 2$, $\lim\limits_{x\to 0}\frac{u(x)}{\log|x|} = 0$.

b. If $K[u]$ is the Kelvin transform of u, find $\Delta(K[u])$ in terms of Δu. Conclude that $K[u]$ is harmonic if and only if u is harmonic. Recall $K[u](x) = |x|^{2-n}u(x^*)$, $x^* = |x|^{-2}x$.

c. Prove that if $n > 2$ and u is harmonic near infinity. Prove that u is harmonic at infinity if and only if $\lim\limits_{|x|\to+\infty} u(x) = 0$. (This was stated as a Lemma in lecture—JC's slides had a typo, but since they were slides, you didn't take notes, so that's not a problem—the slides (corrected!) are available.)

3. If v is harmonic at infinity and $n > 2$, v admits an expansion at infinity in terms of spherical harmonics. We derived the first two terms which give $v(x) = \frac{a_0}{|x|^{n-2}} + \frac{a_i x^i}{|x|^n} + O(|x|^n)$. Derive the next order term, in case $n = 3$.

4. Let $(\mathbb{S}^n, g_0)$ be the standard unit round sphere, $\mathbb{S}^n$ embedded in $\mathbb{R}^{n+1}$ as $\{|x| = 1\}$. It is a fact that the lowest positive eigenvalue λ_1 for Δ_{g_0} corresponds to the eigenfunctions x^i (Euclidean coordinates) restricted to the sphere. Compute $\lambda_1 = n$ by using $\Delta_{g_0}(x^i) = -\lambda_1 x^i$. Multiply by x^i, integrate by parts, and use the fact that $\nabla_{g_0}x^i$ is the tangential component of $\nabla x^i = e_i = \frac{\partial}{\partial x^i}$.

5. Recall Bôcher's Theorem: if $u > 0$ is harmonic in a punctured ball $B\setminus\{0\}$, there exist v harmonic in B and $b \geq 0$ so that $u(x) = \begin{cases} b\log(\frac{1}{|x|}) + v(x)\,, & n = 2 \\ b|x|^{2-n} + v(x)\,, & n > 2. \end{cases}$

a. Show that b and v are uniquely determined.

b. $\Omega \subset \mathbb{R}^n$ is an open set, $n > 2$. If u is harmonic in $\Omega\setminus\{a\}$ ($a \in \Omega$), so that $u > 0$ in a deleted neighborhood of a, show there is a number $b \geq 0$ and a function v harmonic on all of Ω so that on $\Omega\setminus\{a\}$, $u(x) = b|x-a|^{2-n} + v(x)$.

c. $n > 2$. If u is harmonic on $B\setminus\{0\}$, and $\liminf\limits_{x\to 0}|x|^{n-2}u(x) > -\infty$, there exists v harmonic in B, $b \in \mathbb{R}$ so that $u(x) = b|x|^{n-2} + v(x)$ on $B\setminus\{0\}$.