Some exact solutions to the Einstein Equation

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We seek some basic examples of spacetimes $(\bar{M}^4, \bar{g})$ satisfying

$$\mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T.$$

It is natural to start by assuming $\bar{M}^4$ has a simple topology

$$\bar{M}^4 = I \times M^3$$

where I is an open interval and M^3 is a 3-manifold. We also assume the Lorentz metric $\bar{g}$ takes a simple form:

(i) $\bar{g} = -dt^2 + g_t$, where $\{g_t\}$ is a 1-parameter family of Riemannian metrics on M

(ii) $\bar{g} = -N^2 dt^2 + g$, where g is a fixed Riemannian metric on M and $N = N(x)$ is a positive function on M.

Robertson-Walker Spacetimes

We start with assumption (i). The simplest example of a 1-parameter family of metrics $\{g_t\}$ is $g_t = a^2(t)g$ where g is a fixed Riemannian metric and $a > 0$ is a function of t.

Direct calculation shows that the Ricci curvature of $\bar{g} = -dt^2 + a^2 g$ satisfies

- $\mathrm{Ric}(\partial_t, \partial_t) = -\dfrac{3a''}{a}$
- $\mathrm{Ric}(\partial_t, V) = 0, \;\; \forall\ V$ tangent to M
- $\mathrm{Ric}(V, W) = \mathrm{Ric}_g(V, W) + a^2 g(V, W)\left[\dfrac{a''}{a} + 2\dfrac{a'^2}{a^2}\right],$

  $\forall\ V, W$ tangent to M.

Therefore, the scalar curvature $\bar{R}$ of $\bar{g}$ is $\bar{R} = \frac{R}{a^2} + 6\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)$ where R is the scalar curvature of g. The Einstein tensor $G = \mathrm{Ric} - \frac{1}{2}\bar{R}\bar{g}$ now is given by

- $G(\partial_t, \partial_t) = \frac{R}{2a^2} + 3\frac{a'^2}{a^2}$
- $G(\partial_t, V) = 0, \;\; \forall\; V$ tangent to M
- $G(V, W) = \mathrm{Ric}_g(V, W) - a^2 g(V, W)\left[2\frac{a''}{a} + \frac{a'^2}{a^2} + \frac{R}{2a^2}\right]$, $\forall\; V, W$ tangent to M.

To obtain explicit examples, we now assume that (M^3, g) is *a space form*, i.e. $\mathrm{Ric}_g = 2kg$ for some constant k. In this case,

$$G = 3\left(\frac{k}{a^2} + \frac{a'^2}{a^2}\right) dt^2 + \left[-2\frac{a''}{a} - \frac{a'^2}{a^2} - \frac{k}{a^2}\right] g.$$

Hence, $(\bar{M}, \bar{g})$ is a solution to the Einstein Equation $G = 8\pi T$ provided the stress-energy tensor T on $\bar{M}^4 = I \times M^3$ satisfies

$$8\pi T = 3\left(\frac{k}{a^2} + \frac{a'^2}{a^2}\right) dt^2 + \left[-2\frac{a''}{a} - \frac{a'^2}{a^2} - \frac{k}{a^2}\right] g.$$

A stress-energy tensor T on a spacetime $(\bar{M}^4, \bar{g})$ is said to have the form of *a perfect fluid* if there exists a triple (U, ρ, p) where

- U is a future-directed timelike, unit vector field, called the *flow vector field*
- ρ, p are functions on $\bar{M}^4$ (ρ called the *energy density* and p called the *pressure*) such that

$$T = (\rho + p)\omega \otimes \omega + p\bar{g} \tag{1}$$

  where ω is the 1-form dual to U.

Definition A spacetime $(\bar{M}^4, \bar{g}) = (I \times M^3, -dt^2 + a^2(t)g)$ is called a *Robertson-Walker spacetime* if (M^3, g) is a connected, Riemannian manifold of constant sectional curvature and $a(t)$ is a positive function on an open interval I.

Proposition *A Robertson-Walker spacetime $(\bar{M}^4, \bar{g})$ is a solution to the Einstein Equation $G = 8\pi T$ with T having the form of a perfect fluid*

$$T = (\rho + p)dt \otimes dt + p\bar{g}$$

if and only if

$$8\pi\rho = 3\left(\frac{k}{a^2} + \frac{a'^2}{a^2}\right), \quad -8\pi p = 2\frac{a''}{a} + \frac{a'^2}{a^2} + \frac{k}{a^2}. \tag{2}$$

Note that (2) implies $\frac{a''}{a} = (-1)\frac{4}{3}\pi(\rho + 3p), \rho' = (-3)\frac{a'}{a}(\rho + p)$.
Analytically, one would like to draw conclusion on $a(t)$ assuming

$$\rho + 3p \geq 0 \;\; \text{and} \;\; \rho + p \geq 0. \tag{3}$$

Physically, (3) translates into an *Energy Condition* on T: We say a stress-energy tensor T satisfies the *strong energy condition* if

$$T(V, V) \geq \frac{1}{2}(\operatorname{tr} T)\langle V, V\rangle, \; \forall \; \textit{timelike} \; V. \tag{4}$$

Assuming $G = 8\pi T$ holds, one checks that (4) is equivalent to

$$\operatorname{Ric}(V, V) \geq 0, \; \forall \; \textit{timelike} \; V \tag{5}$$

which is often referred as the *timelike convergence condition*.

Proposition *Suppose a Robertson-Walker spacetime* $(\bar{M}^4, \bar{g})$ *is a perfect fluid solution to the Einstein Equation* $G = 8\pi T$ *where*

$$T = (p + \rho) dt \otimes dt + p\bar{g}$$

satisfies the strong energy condition. If $H(t_0) = \frac{a'(t_0)}{a(t_0)} > 0$ *for some* $t_0 \in I$*, then*

- I *has a finite left endpoint* $t_{initial}$ *with* $t_0 - H_0^{-1} < t_{initial} < t_0$
- $a'(t) > 0$ *on* $(t_{initial}, t_0)$ *and* $\rho'(t) \leq 0$ *on* $(t_{initial}, t_0)$
- *every timelike geodesic normal to the slice* $\{t\} \times M^3$ *is past incomplete.*

Remark: If $\lim_{t \to t_{initial}} a(t) = 0$ and $\lim_{t \to t_{initial}} a'(t) = \infty$, $t_{initial}$ is called a *big bang*. In this case, one checks $\lim_{t \to t_{initial}} \rho(t) = \infty$.

A *dust* is a perfect fluid with $p = 0$ and $\rho > 0$.

Exercise *Find the Robertson-Walker spacetimes that are dust solutions to the Einstein Equation.*

Schwarzschild Spacetime

We proceed to assumption (2), i.e.

$$\bar{g} = -N^2 dt^2 + g \ \text{ on } \bar{M}^4 = I \times M^3$$

where g is a fixed Riemannian metric on M and $N = N(x)$ is a positive function on M.

In this case, the Ricci curvature of $\bar{g} = -N^2 dt^2 + g$ satisfies

- $\mathrm{Ric}(\partial_t, \partial_t) = N\Delta N$
- $\mathrm{Ric}(\partial_t, V) = 0, \ \forall \ V \in TM$
- $\mathrm{Ric}(V, W) = \mathrm{Ric}_g(V, W) - \dfrac{1}{N}\nabla^2 N(V, W), \ \forall \ V, W \in TM.$

Here $\triangle$ and ∇^2 are the Laplacian and Hessian w.r.t g on M^3.

We consider the Vacuum Einstein Equation imposed on $\bar{g}$, i.e.

$$\mathrm{Ric}(\bar{g}) - \frac{1}{2}\bar{R}\bar{g} = 0 \Leftrightarrow \mathrm{Ric}(\bar{g}) = 0,$$

which in terms of N and g becomes

$$\left\{ \begin{array}{ccc} \Delta N & = & 0 \\ N\mathrm{Ric}_g - \nabla^2 N & = & 0. \end{array} \right. \tag{6}$$

Taking trace of (6) gives $R(g) = 0$, where $R(g)$ is the scalar curvature of g.

In what follows, we seek a *rotationally symmetric* (M^3, g) that is a solution to (6) with a *rotationally symmetric* N.

Exercise *g is rotationally symmetric $\Rightarrow$ g is conformally flat.*

Thus, it is natural to seek those $g = u^4(\rho) g_E$ where $g_E = d\rho^2 + \rho^2 g_{\mathbb{S}^2}$ is the Euclidean metric.

Now $R(\bar{g}) = 0 \Rightarrow \Delta_{g_E} u(\rho) = 0 \Rightarrow u(\rho) = A + \dfrac{B}{\rho}$ where A, B are constants.

When $A = 0$, $g = \left(\dfrac{1}{\rho}\right)^4 g_E$ is flat. So we may assume $A \neq 0$. By rescaling g_E, we assume $A = 1$ and write $B = \dfrac{\mathfrak{m}}{2}$,

$$u = 1 + \frac{\mathfrak{m}}{2\rho}, \quad \textit{and } g = \left(1 + \frac{\mathfrak{m}}{2\rho}\right)^4 (d\rho^2 + \rho^2 g_{\mathbb{S}^2}).$$

We want to write g in the usual rotationally symmetric form

$$g = h(r)dr^2 + r^2 g_{\mathbb{S}^2}.$$

Making a change of variable $r = \rho\left(1 + \frac{\mathfrak{m}}{2\rho}\right)^2$, we have

$$g = \left(1 - \frac{2\mathfrak{m}}{r}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2}. \tag{7}$$

For $N = N(r)$, it is determined by $\Delta N = 0$.

<u>Exercise</u> *Verify that* $N(r) = \sqrt{1 - \frac{2\mathfrak{m}}{r}}$ *is a* g*-harmonic function. Moreover,* N *and* g *together satisfy* (6).

Thus, we see that

$$\bar{g}_{\mathfrak{m}} = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^2}, \ \textit{where } h(r) = 1 - \frac{2\mathfrak{m}}{r} \quad (8)$$

satisfies the Vacuum Einstein Equation $\mathrm{Ric}(\bar{g}) = 0$.

When $\mathfrak{m} = 0$, $h(r) = 1$, $\bar{g}_{\mathfrak{m}}$ is the Minkowski spacetime.

Suppose $\mathfrak{m} < 0$. Consider the Riemannian metric $g = h(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^2}$ where $r \in (0, \infty)$. Let $\nu = h^{\frac{1}{2}}\partial_r$ with $|\nu|_g = 1$, one computes $Ric(\nu, \nu) = -\dfrac{2\mathfrak{m}}{r^3}$, which shows that $r = 0$ is a singularity for g, hence a singularity for $\bar{g}$ as $\{t = constant\}$ is totally geodesic in $\bar{g}$.

For most of our discussion, we will assume $\mathfrak{m} > 0$.

Consider the future-directed timelike vector field $V = h^{-\frac{1}{2}}\partial_t$. The integral curves of V are called *Schwarzschild observers*. Let γ be one of such an observers.

<u>Exercise</u>: $\gamma'' = D_V V = \dfrac{\mathfrak{m}}{r^2}\partial_r$.

Thus, γ is not freely falling unless $\mathfrak{m} = 0$. Heuristically, the "external force" $F_{external} = \dfrac{\mathfrak{m}}{r^2}\partial_r$ on γ should balance the effect of gravity. In the Newtonian world, this would say the gravitational force exerted on γ is $F_{gravity} = -\dfrac{\mathfrak{m}}{r^2}\partial_r$, suggesting that the parameter $\mathfrak{m}$ be called the *mass* of the the Schwarschild spacetime.