presenter: Dana Randall

title: Biased permutations

problem:

Consider the following nearest neighbor Markov chain to sample from the set of permutations on n objects $\{1, \ldots, n\}$. We are given set of input parameters $\mathbf{P} = \{p_{x,y}\}$ with $1/2 \leq p_{x,y} \leq 1$. At each point in time, the Markov chain $\mathcal{M}_{nn}$ picks a pair of adjacent elements in the current permutation, say x and y in positions $i-1$ and i. If $x < y$ then we put x in front of y with probability $p_{x,y}$ and we put y in front of x with probability $1 - p_{x,y}$. We are interested in understanding for which choices of $\mathbf{P}$ the chain is rapidly mixing (i.e., converging to equilibrium in polynomial time). Note, it is not difficult to see that without the restriction that $1/2 \leq p_{x,y}$ for all x, y the chain might be slow to converge to equilibrium.

Jim Fill conjectured that if $\mathbf{P}$ also satisfies a "regularity condition" requiring the $p_{x,y}$ to be concave in x and in y then the spectral gap is maximized when $p_{x,y} = 1/2$ for all x and y. The mixing rate of $\mathcal{M}_{nn}$ is only known in special cases. When $p_{x,y} = 1/2$ for all x, y, the chain is known to mix in time $\Theta(n^3 \log n)$ [1, 3, 5]. When $p_{x,y} = q$ for some $1 < q < 1/2$, then the chain is known to mix in time $\Theta(n^2)$ [2].

In recent work with Sarah Miracle and Amanda Streib [4], we show the chain is always rapidly mixing in two cases: For the first class, we are given $r_1, \ldots, r_{n-1}$ and $p_{x,y} = r_x$ if $x < y$, so the probability of an inversion depends only on the object with higher (or lower) rank. In the second class, we are given parameters $1/2 \leq q_1, \ldots, q_{\log n} \leq 1$. Then for $x < y$, we define $p_{x,y} = q_i$ where i is the highest order bit in which the binary representations of x and y differ and as above $p_{y,x} = 1 - p_{x,y}$. In other words, if $n = 2^k$, then $p_{x,y} = q_1$ if $x \leq n/2$ and $y > n/2$. However, if $x \leq n/4$ and $n/4 < y \leq n/2$, then they have rate $p_{x,y} = q_2$.

Unfortunately, we also found a counterexample that disproves the conjecture in the most general setting. Suppose, that we are sampling permutations with n entries, for n even. Let $M = n^{2/3}$, $0 < \delta < 1/2$ be a constant, $\epsilon = 1/n^2$. For $i < j \leq n/2$ or $n/2 < i < j$, $p_{i,j} = 1$, ensuring that once the elements $1, 2, ..., n/2$ get in order, they stay in order (and similarly for the elements $n/2+1, n/2+2, ..., n$). The $p_{i,j}$ values for $i \leq n/2 < j$ are defined so that $p_{i,j} = 1 - \delta$ if $i - j + n + 1 \leq n/2 + M$, and $p_{i,j} = 1/2 + \epsilon$ otherwise. The intuition is that we can interpret each permutation as a lattice path from $(0, n)$ to $(n, 0)$ that always goes to the right or down; if we encounter an element from the first half of the "deck" we go right and from the second half we go down. The $p_{i,j}$ are chosen so that the configurations with the highest stationary probability come close to the point (n, n), but we are unlikely to reach these in polynomial time because typical paths will stay close to the diagonal connecting $(0, n)$ and $(n, 0)$.

The open problem is to find more general classes of positively biased permutations for which the chain is rapidly mixing. For example, is the spectral gap is always bounded when $\mathbf{P}$ also satisfies the regularity condition (or something similar)?

References

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