A rapid course in Morse homology

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- For $a < \min f$ we have $X^a = \emptyset$, and for $a > \max f$ we have $X^a = X$.
- To understand X is suffices to under stand how X^a changes as a increases from $\min f$ to $\max f$

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- The set of critical values of f has measure zero. (Sard's Theorem)

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To proceed we restrict ourselves to the class of Morse functions.

Recall

Lemma (Morse Lemma)

If p is a nondegenerate critical point of $f : X \to \mathbb{R}$ then there are coordinates $(x_1, \ldots, x_d)$ defined near p for which

$$H(x_1, \ldots, x_d) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_d^2.$$

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Example

If p is a local max (min) of f, then its Morse index is d (0).

Theorem (Morse)

Let p be a critical point of a Morse function f with Morse index $\lambda(p)$. Set $c = f(p)$ and choose ϵ sufficiently small so that c is the only critical value in $[c-\epsilon, c+\epsilon]$. The manifold $X^{c+\epsilon}$ has the homotopy type of the manifold $X^{c-\epsilon}$ with a $\lambda(p)$-cell attached.

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Example

Suppose that f attains its minimum value, c, at a unique point p. Then for sufficiently small $\epsilon > 0$ the manifolds $X^{c+\epsilon}$ have the homotopy type of a 0-cell. In fact, it follows immediately from the Morse Lemma that these $X^{c+\epsilon}$ are diffeomorphic to the closed d-ball.

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Morse homology is a construction which allows one to recover this topological information captured by f in terms of the dynamics of negative gradient vector fields. Roughly speaking, it gives a model for the homology of the CW-complex above and hence for the homology of X.

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- Note

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0}(f(\psi_t(p))) &= df(-\nabla f(p)) \\ &= -g(\nabla f(p), \nabla f)(p) \\ &\leq 0 \end{aligned}$$

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  with equality iff p is a critical point of f.
- For all $p \in X$ we then have

$$\lim_{t \to \pm\infty} \psi_t(p) \in \mathcal{C}(f).$$

For $p \in \mathcal{C}(f)$ set

- $W^u(p) = \{r \in X \mid \lim_{s\to-\infty} \psi_t(r) = p\}$

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Each $W^u(p)$ is an embedded copy of $\mathbb{R}^{\lambda(p)}$, and each $W^s(p)$ is an embedded copy of $\mathbb{R}^{d-\lambda(p)}$.

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$$X = \bigcup_{p \in C(f)} W^u(p)$$

Morse Chain Groups

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Let $\mathrm{CM}_k(f)$ be the $Z_2 = \mathbb{Z}/2\mathbb{Z}$-vectorspace generated by the elements of $C_k(f)$.

If $\mathcal{C}_k(f) = \{p_1, \ldots, p_n\}$ then $\mathrm{CM}_k(f)$ is the vector space of linear combinations of the form $\sum_{j=1}^n c_j p_j$ where $c_j \in \mathbb{Z}_2$.

The Morse boundary operator

Set $\mu(p,q) = W^u(p) \cap W^s(q)$

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Note

$$\begin{array}{rcl} W^u(p) \cap W^s(q) & = & \left\{ \gamma \colon \mathbb{R} \to X \mid \begin{array}{c} \dot{\gamma}(t) = -\nabla f(\gamma(t)) \\ \gamma(-\infty) = p \\ \gamma(+\infty) = q \end{array} \right\} \\ r & \leftrightarrow & \gamma(0) \end{array}$$

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i.e. $\mu(p,q)$ is a set of negative gradient trajectories.

Properties of $\mu(x, y)$: Transversality

Theorem (Transversality)

For a generic choice of g we have $W^u(p) \pitchfork W^s(q)$. In this case each $\mu(p, q)$ is a smooth submanifold of dimension $\lambda(p) + (d - \lambda(q)) - d = \lambda(p) - \lambda(q)$.

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Since ∇f is a time dependent vector field there is a free $\mathbb{R}$-action on each $\mu(p, q)$ with $p \neq q$, i.e.,

$$\tau \cdot r = \psi_\tau(r) \text{ or } \tau \cdot \gamma(t) = \gamma(t + \tau)$$

Set $\widehat{\mu}(p, q) = \mu(p, q)/\mathbb{R}$.

The Morse boundary continued.

- For an element $p \in \mathcal{C}_k(f)$ set

$$\partial_k(p) = \sum_{q \in \mathcal{C}_{k-1}(f)} \#_2 \widehat{\mu}(p,q) q,$$

where $\#_2 \widehat{\mu}(p,q)$ is the number of elements modulo two in the zero dimensional manifold $\widehat{\mu}(p,q)$.

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where $\#_2 \widehat{\mu}(p,q)$ is the number of elements modulo two in the zero dimensional manifold $\widehat{\mu}(p,q)$.
- Let $\partial_k \colon \mathrm{CM}_k(f) \to \mathrm{CM}_{k-1}(f)$ be the linear extension of this map, i.e.,
$$\partial_k \left(\sum_{j=1}^{n} c_j p_j \right) = \sum_{j=1}^{n} c_j \partial_k(p_j).$$

The Morse boundary continued.

Define the Morse boundary operator $\partial\colon \mathrm{CM}(f) \to \mathrm{CM}(f)$ by setting

$$\partial = \sum_{k=0}^{d} \partial_k .$$

Theorem (Boundary Theorem)

The map ∂ is well-defined and satisfies $\partial \circ \partial = 0$.

Compactness and Gluing for $\widehat{\mu}(p, q)$

Definition

A k-**times broken** negative gradient trajectory from p to q is a collection of $k+1$ classes $\gamma_j \in \widehat{\mu}(r_j, r_{j+1})$ such that $r_0 = p$, $r_{k+1} = q$ and

$$\lambda(p) > \lambda(r_1) > \cdots > \lambda(r_k) > \lambda(q).$$

Theorem

The moduli space $\widehat{\mu}(p, q)$ has a natural compactification as a smooth $(\lambda(p) - \lambda(q) - 1)$-dimensional manifold with corners. The strata of codimension k in the compactification is the set of k-times broken negative gradient trajectory from p to q.

Compactness and Gluing for trajectories of $-\nabla f$

Example

If $\lambda(p) = \lambda(q) + 1$ then $\widehat{\mu}(p, q)$ is a compact 0-dimensional manifold.

In particular, it is not possible to have a broken trajectory from p to q since there are no indices between $\lambda(p)$ and $\lambda(q)$.
This immediately implies that

$$\partial_k(p) = \sum_{q \in \mathcal{C}_{k-1}(f)} \#_2 \widehat{\mu}(p, q) q,$$

is well-defined, and hence that ∂ is well-defined.

Compactness and Gluing for trajectories of $-\nabla f$

The second part of the Boundary theorem will follow from

Example

If $\lambda(p) = \lambda(q) + 2$ then the compactification of $\widehat{\mu}(p, q)$ is a 1-dimensional manifold with boundary given by

$$\bigcup_{r \in \mathcal{C}_{\lambda(p)-1}(f)} \widehat{\mu}(p, r) \times \widehat{\mu}(r, q).$$

We will also need the familiar fact that every (possibly disconnected) compact 1-manifold is a finite collection of circles and closed intervals. In particular, the number of its boundary components is even.

$$\begin{aligned}
\partial \circ \partial(p) &= \partial\left(\sum_{r \in \mathcal{C}_{k-1}(f)} \#_2 \widehat{\mu}(p,r) r\right) \\
&= \sum_{r \in \mathcal{C}_{k-1}(f)} \#_2 \widehat{\mu}(p,r) \left(\sum_{q \in \mathcal{C}_{k-2}(f)} \#_2 \widehat{\mu}(r,q) q\right) \\
&= \sum_{q \in \mathcal{C}_{k-2}(f)} \left(\sum_{r \in \mathcal{C}_{k-1}(f)} \#_2 \widehat{\mu}(p,r) \#_2 \widehat{\mu}(r,q)\right) q \\
&= \sum_{q \in \mathcal{C}_{k-2}(f)} \#_2 \left(\bigcup_{r \in \mathcal{C}_{k-1}(f)} \widehat{\mu}(p,r) \times \widehat{\mu}(r,q)\right) q \\
&= 0.
\end{aligned}$$

Morse Homology

We can now define the Morse homology groups for $k = 0, \ldots, d$:

$$\mathrm{HM}_k(f, g) = \frac{\ker \partial_k}{\partial_{k+1}(\mathrm{CM}_{k+1}(f))}$$

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Theorem (Thom, Smale, Witten)

Each group $\mathrm{HM}_k(f, g)$ *does not depend the choices of* f *and* g*. In fact,*

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Corollary (Weak Morse inequalities)

If b_k *is the rank of* $\mathrm{H}_k(X; \mathbb{Z}_2)$ *then every Morse function* f *on* X *has at least* b_k *critical points of index* k *for* $k = 1, \dots, d$.

Continuation maps

Let (f_0, g_0) and (f_1, g_1) be two pairs of Morse data.

We now construct a chain map

$$\sigma_{01} \colon \mathrm{CM}_k(f_0, g_0) \longrightarrow \mathrm{CM}_k(f_1, g_1)$$

which induces the identity map in homology.

Continuation maps

Let f_s be a family of functions for $s \in \mathbb{R}$ such that $f_s = f_0$ when s is sufficiently negative, and $f_s = f_1$ when s is sufficiently positive.
Let g_s be a similar family of metrics from g_0 to g_1.
For $p \in \mathcal{C}(f_0)$ and $q \in \mathcal{C}(f_1)$ set

$$\mu_s(p,q) = \left\{ \gamma\colon \mathbb{R} \to X \mid \begin{array}{c} \dot\gamma(s) = -\nabla f_s(\gamma(s)) \\ \gamma(-\infty) = p \\ \gamma(+\infty) = q \end{array} \right\}$$

where ∇f_s is the gradient of f_s w.r.t. g_s.

Continuation maps

Theorem (Transversality)

For a generic choice of the pair (f_s, g_s) the spaces $\mu_s(p,q)$ are smooth manifolds of dimension $\lambda(p) - \lambda(q)$.

For $p \in \mathcal{C}_k(f_0)$ set

$$\sigma_{01}(p) = \sum_{q \in \mathcal{C}_k(f_1)} \#_2(\mu_s(p,q))q$$

This is a chain map.

Continuation maps

The induced map in homology has the following properties:

1. $\sigma_{01} \colon \mathrm{HM}_k(f_0, g_0) \to \mathrm{HM}_k(f_1, g_1)$ does not depend on the choice of (f_s, g_s)
2. $\sigma_{12} \circ \sigma_{01} = \sigma_{02}$
3. σ_{00} is the identity.

It follows easily from these properties that $\sigma_{10} \circ \sigma_{01}$ and $\sigma_{01} \circ \sigma_{10}$ are the identity maps on $\mathrm{HM}_k(f_0, g_0)$ and $\mathrm{HM}_k(f_1, g_1)$, respectively. Hence σ_{01} is both 1-1 and onto.