Using Math to Teach Math

Mathematicians and educators
investigate the mathematics
needed for teaching

Mark Hoover Thames

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Foreword

In 2004, the Mathematical Sciences Research Institute (MSRI) launched a workshop series, *Critical Issues in Mathematics Education*, to provide opportunities for mathematicians to cooperate with experts from other communities on the improvement of mathematics teaching and learning. In designing and hosting these conferences, MSRI seeks to legitimize such cooperation and to lend support for interdisciplinary progress on critical issues in mathematics education.

The second workshop in the series, *Mathematical Knowledge for Teaching (K-8): Why, What, and How?*, was held at the Asilomar Conference Grounds, Pacific Grove, CA, May 25-28, 2005. The focus on mathematical knowledge for teaching was chosen due to broad consensus that teacher knowledge of mathematics is fundamental to quality instruction. The premise of the workshop was that improving students' mathematics learning in the U.S. depends on improving mathematics teaching, for which teacher knowledge of mathematics is a key factor. The workshop brought together different groups for whom the issue of teacher mathematical knowledge is of critical concern and explored current perspectives, evidence, and programs. Three questions structured its interactive design:

1. Why should K-8 teachers know mathematics?
2. What is the nature of the knowledge of mathematics needed for effective teaching?
3. What can mathematics departments and schools of education do to help teachers develop such knowledge?

The goal of the present document, commissioned by MSRI, is to draw more mathematicians' attention to the problem of the mathematical preparation of teachers and to assist those who want to get involved. It is my hope that by providing a coherent synthesis of the many ideas assembled at the workshop, this document will support mathematicians, mathematics educators, and others in their efforts to improve mathematics courses and programs for K-8 teachers.

The organization of the workshop and of this booklet draws significantly from research conducted at the University of Michigan by Deborah Ball, Hyman Bass, Heather Hill, myself, and others. Our work seeks to understand the nature of teacher mathematical knowledge as it arises out of, and is used in, teaching. Emerging from this research is a characterization of mathematical knowledge for teaching as, just that, mathematical knowledge *for teaching*. In other words, the mathematics teachers need to know is connected to the distinctive work teachers do. This characterization of *mathematical knowledge for teaching* focuses attention on what matters most — that the mathematics taught to teachers be useful to them and help to improve teaching and learning. It also provides a framework for viewing and connecting a variety of efforts across the country that work on the problem of teacher content knowledge in different ways. For both reasons, this characterization of teachers' mathematical knowledge served well as the basic structure for the workshop and for this document.

In drafting this booklet, I have made liberal use of ideas developed by the research group at the University of Michigan and those presented at the workshop, especially in talks given by David Monk, Heather Hill, James Hiebert, Roger Howe, Liping Ma, Hyman Bass, Randy Philipp, Robert Moses, Jill Adler, Marta Civil, and Lena Licón Khisty. I gratefully acknowledges these mathematicians, mathematics educators, educational researchers, and teachers, as well as those who provided feedback on a draft: the Workshop Organizing Committee, associates of MSRI, and numerous workshop presenters and attendees. I also want to thank Erin Dahlbeck at Goodby, Silverstein & Partners for assistance with the design and layout of the document and Laurie Sleep, Darcia Harris Bowman, and Chad Bowman for their insightful editing. The errors and inadequacies, undoubtably, remain mine.

Mark Thames

Contents

What's the problem with teachers' mathematical knowledge

and whose problem is it?

One might think that success in school as a child is adequate mathematical preparation for K-8 teaching. Evidence, however, suggests that success in school does not guarantee that one knows a subject well. More to the point, such success does not necessarily mean a teacher knows the subject well enough to help others learn it. While most teachers liked school as children and were relatively successful, many lack important mathematical knowledge needed for effective teaching.

The mathematical demands of K-8 teaching are quite substantial. In addition to knowing the mathematical definitions of terms, teachers must be able to use definitions effectively when teaching. In defining a term, they need to be able to find language that is meaningful to children, yet mathematically correct.

Defining an even number to be *an integer multiple of 2* is of little use if students do not know the terms *integer* or *multiple*. Defining an even number to be *a whole number that is two times another number,* while perhaps accessible to children, admits all whole numbers (e.g., 1 is two times one half). Amending this definition to say, *two times another whole number,* may seem to solve the problem, but this definition excludes negative multiples of 2. Later, when students learn about negative integers, they should not have to unlearn an earlier definition. Changing the definition to *a whole number is even if it is 2 times another whole number* is mathematically honest, yet respects the scope of children's experience.

These subtle and mathematically demanding issues would challenge most college students, even mathematics majors. Classroom teachers, however, contend regularly with mathematical problems of exactly this type.

Teachers must routinely:

- give clear explanations,
- choose useful examples,
- evaluate students' ideas,
- select appropriate representations,
- modify problems to be easier or harder,
- recognize different ways to solve the same problem,
- explain goals and mathematical purposes to others, and
- build correspondences between models and procedures.

These are relatively uncommon skills in our society and are rarely taught. They require significant mathematical knowledge not typically needed by people who do not teach.[1] Yet teachers need to be proficient at these tasks amidst the busy flow of classroom life. For instance, anyone who knows the content should be able to determine that a pupil has produced a wrong answer to a problem. But figuring out what a student did wrong—spotting the method and guessing its rationale—requires greater skill and knowledge. Likewise, teachers must recognize when a "right" answer is the result of faulty thinking. These analyses represent mathematically demanding activities specifically required of those who teach. Mathematics of this kind is mathematical knowledge for teaching. As such, it is unique to the profession of teaching and is distinct as a body of knowledge within the field of mathematics. But where and when do teachers learn such mathematics?

Throughout this booklet are "Mathematics for Teaching" boxes with questions adapted from a project at the University of Michigan. These are mathematics questions, but represent the mathematical problems teachers face in their daily work.

Mathematics for Teaching

Develop a definition of polygon that is:

1. Mathematically correct, and

2. Usable by fifth graders (namely, using only terms that can be made understandable to students at this level).

David Eisenbud, MSRI

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1   Throughout this document, the term knowledge is used broadly to include the knowledge, skill, dispositions, and habits of mind that support doing mathematics.

The Role of Mathematics Departments in Preparing Teachers

Through the first half of the 20th century, it was considered sufficient for elementary teachers to know mathematics a few grades beyond the content they were teaching. Indeed, the mathematical preparation of elementary teachers was not a concern of colleges and universities nationally until the 1960s (Ferrini-Mundy & Graham, 2003). In 1960, the Mathematical Association of America's Panel on Teacher Training recommended:

• one course on the basic concepts of algebra, and

• one course on informal geometry.

These courses were meant to duplicate material studied in high school "from a more sophisticated college-level point of view" (MAA, 1960, p. 634). However, if courses for teachers merely revisit what was taught before, should they carry college credit? In what sense does thorough treatment in a college course differ from what is taught in schools? In what sense is it more?

Different ideas were proposed for what these courses ought to include, such as opportunities to develop greater mathematical sophistication, exposure to core ideas of the discipline, or advanced study of topics related to the elementary school curriculum. Unfortunately, though, the content of courses for teachers was only logically reasoned, not empirically investigated. In other words, little was done to explore what mathematics would make a difference for teachers or for the improvement of student learning. The result has been courses that leave teachers under-prepared.

Textbook by Parker and Baldridge (2004)
with student textbooks from Singapore

U.S. teachers take most of their mathematics courses in mathematics departments. These courses are mostly characterized by unhappy students in classes associated with second-class citizenship for instructors. Course content is more inherited and haphazard than logically coherent or empirically verified. These courses ought to equip teachers with the knowledge and skills they will need for their work in K-8 classrooms. Too often, however, prospective teachers see the courses as irrelevant to their work, mathematicians think they lack substance, and the public sees them as having minimal benefit. Not surprisingly, there is some truth in all these views. Without a clear sense of what constitutes mathematically rigorous content relevant to teaching and learning, these courses are bound to disappoint all parties.

The remainder of this booklet examines:

1. *The evidence that mathematical knowledge for teaching matters for student learning*

2. *The nature and extent of the mathematical knowledge needed by teachers, and*

3. *Ways for mathematicians to get involved.*

What are mathematicians doing to improve the mathematical preparation of teachers?

Mathematicians who teach mathematics courses for K-8 teachers want their courses to improve what happens in the nation's classrooms. That is, mathematicians hope to improve K-8 student learning. However, the link between the college mathematics courses teachers take and classroom learning is not always immediate. With this in mind, a number of mathematicians are striving to make the link more apparent. They work to improve textbooks, courses, and programs for teachers by combining an intellectually honest handling of mathematics with a respectful consideration of the demands of teaching.

The grey sidebars in this document and many of the accompanying pictures feature mathematicians who made presentations at the May 2005 workshop. These are included as examples of what mathematicians might contribute.

Mathematics for Teaching

Students produced the following incorrect answers for the product of 49 and 25. In each case, what process plausibly produces this answer? What might have led a student to do this?

(a)	(b)	(c)
49	49	49
x 25	x 25	x 25
405	225	1250
108	100	25
1485	325	1275

What's the evidence
that teacher mathematical knowledge matters?

We are getting a clearer sense that certain mathematically rigorous content is relevant to teaching. Proving it is another matter. Establishing empirical evidence for a claim about the effects of teacher knowledge requires the consideration of multiple studies and varied perspectives. Its synthesis also requires a healthy dose of practical judgment.
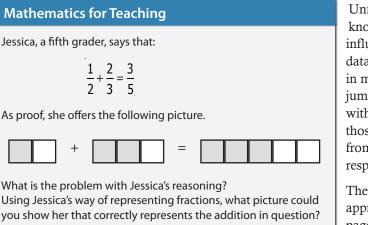
A concerted effort to find evidence for the effects of teacher knowledge on student achievement extends back to at least Begle (1979). This research found that most of the studies conducted during the middle of the 20th century showed negligible effects of the mathematical preparation of teachers on student achievement. Nearly as many reported small negative effects on student achievement as reported small positive effects. This was discouraging news, but led many to think that the problem was how the mathematical knowledge of teachers was being measured. Does choice of major influence student achievemen? The number of college mathematics courses taken? One's score on a standardized test?

The problem was more than one of proxies for measuring mathematical knowledge. Inadequate measures, concepts, models, and methods have limited researchers' ability to determine whether, and to what degree, improving the mathematical knowledge of teachers improves student achievement. Studies by economists and researchers of teaching, plus data from international achievement studies, suggest, in different ways, the importance of understanding how teacher knowledge effects student achievement.

Deborah Ball, University of Michigan, and
Jim Milgram, Stanford University

Teacher Knowledge → Teaching → Student Achievement

Implicit in most people's thinking is the model above suggesting how *teacher knowledge* influences *teaching* in ways that improve *student achievement*.

Mathematics for Teaching

Jessica, a fifth grader, says that:

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$$

As proof, she offers the following picture.

☐☐ + ☐☐☐ = ☐☐☐☐☐

What is the problem with Jessica's reasoning?
Using Jessica's way of representing fractions, what picture could you show her that correctly represents the addition in question?

Adapted from Beckmann (2003,).

Unfortunately, our understanding of how teacher knowledge influences teaching and how teaching influences student achievement is inadequate. And most data we have do not address all three of these components in much detail. Too often, people draw conclusions that jump from teacher knowledge to student achievement without considering the role of teaching in bringing about those changes. Or they work backwards, simply inferring from differences in student achievement the factors responsible for improvement.

The following pages provide a brief tour of three approaches to the issue of evidence. The purpose of these pages is twofold. First, different disciplines develop and substantiate arguments in different ways. This overview of research from various points of view shows the types of questions different researchers ask and the methods they use for studying this issue. The second purpose is to size up the evidence, however inconclusive, identifying the best potential answers.

What We're Learning from Economists[2]

Economists have studied education as a system producing certain outcomes associated with certain costs. Under this model, the presumption is that decision makers within schools try to maximize outputs (e.g., student achievement) from any given set of inputs (e.g., per-pupil funding). These "production-function" studies contribute appreciably to our understanding of the factors influencing student achievement.

Before looking at results related to teacher knowledge, consider the following snapshot of the kinds of questions economists are inclined and disinclined to think about.

Typical Issues Economists Think About

- *Containing costs:* Whereas many educators think about education reform with little regard for cost, economists typically focus on cost.

- *Substitutions among inputs:* What happens when we trade one input for another, such as mathematics knowledge for "smartness"? Does teacher mathematics knowledge make up for poor performance on tests of general knowledge? And, for teachers who have general knowledge, does a little content go a long way?

- *Interactions:* Do certain inputs reinforce each other, such that results are greater in combination, or is the opposite true? For instance, does teacher mathematics knowledge interact with years of teaching? Does it interact with the characteristics of the students they teach?

- *Non-linearities:* Think of the law of diminishing returns. Where do resource allocations make the most sense? Are the gains from knowing a little more mathematics greater for teachers who know little, a moderate amount, or a great deal of mathematics?

- *Temporal influence:* Does mathematics knowledge become obsolete over time? Or, if you have no choice but to deal with a teacher with poor mathematics knowledge, during which grades might exposing students to this teacher occasion the least amount of "damage"?

Issues Often Spurned by Economists

- *Motivation:* Economists are interested in behavior, but they are less interested in why people behave the way they do and are skeptical of what people say.

- *Defining Outcomes:* Economists are inclined to let psychometricians and others decide what the outcome or the measure of the outcome should be.

- *Detail:* Economists are comfortable with isolating particular aspects of a complex phenomenon such as teaching and they have confidence that well-understood individual pieces will add up to more comprehensive understandings. They believe detailed descriptions of processes, however, are expensive and best left to others.

While the issues of most interest to economists remain mostly unanswered, one clear finding is that student family background (as measured by socioeconomic status) is consistently a significant predictor of student achievement. The disappointing news is that, beyond this, little is straightforward.

In short, the production-function research suggests that variables in education can be hard to measure, and relationships between inputs and outputs are complex and require subtle analysis and attention to details. The question for an economist becomes this: At what point does digging into the details become too expensive and uninformative?

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2   Ideas in this section are adapted from a presentation given by David Monk.

Writing textbooks for teachers— And using textbooks for children

A handful of mathematicians are writing textbooks that take seriously the practical realities of school teaching. These efforts begin to use our increasing understanding of the mathematical demands of teaching to map out a curriculum of mathematics for teaching.

Ira Papick, University of Missouri-Columbia

Ira Papick (University of Missouri, Columbia) has won numerous teaching awards. With colleagues in mathematics and education, he writes textbooks for middle school teachers and develops courses that use the analysis of middle school textbooks as a central task in those courses. In addition, he teaches courses for prospective middle-grade mathematics teachers and investigates K-12 mathematics curricula — issues of design, analysis, implementation, evaluation, and the curriculum's use to support teacher learning.

http://www.math.missouri.edu/personnel/faculty/papicki.html

Does the production-function literature tell us anything specific about the role of teacher knowledge of mathematics in classroom learning?

Most production-function studies use general measures of teacher knowledge, such as aptitude or verbal skill. Some use proxies for subject matter preparation, such as the number of college mathematics courses taken or whether teachers majored in mathematics. Though most of these studies show small positive effects on student achievement, the results are mixed and the effects for general knowledge and for subject matter preparation are similar.

David Monk, Penn State University

In one such study, Monk (1994) modeled data from the Longitudinal Study of American Youth for both mathematics and science using gains on the National Assessment of Educational Progress (NAEP) as the dependent variable. He reached three conclusions that help to explain the inconsistent results in the literature.

First, he found *diminishing returns*. In this study, Monk discovered that, for teachers of high school juniors, the number of college content courses teachers take makes a difference in the achievement of their students — but only up to a point. After five courses in mathematics and four courses in the physical sciences, additional courses had little effect on student outcomes. (A number of studies have reported no

Sybilla Beckmann, University of Georgia, with her middle school students

effects, or even negative effects, for advanced study of subject matter — as if advanced study creates compressed forms of knowledge that distances teachers from the content they teach and from the struggles of their students.)

Complementing this conclusion is a second one: *interactions* occur among the subject matter studied by teachers, the subject matter they teach, and the kind of students in their classes. For example, positive effects of college mathematics coursework were greater for teachers of juniors than for teachers of sophomores. Likewise, the dynamics between a teacher's college coursework and the learning of that teacher's students depended on the specific high school course: life sciences, physical sciences, advanced mathematics, or remedial mathematics. The dynamics of the effects of mathematics courses on science teaching were also complex.

Sybilla Beckmann (University of Georgia) teaches courses for prospective elementary teachers and has written a textbook. The book starts with the premise that teachers should be keenly aware of the key principles and ideas that underlie the mathematics they teach. Further, teachers should be able to explain the rationale for procedures or formulas they teach, such as why we multiply by the reciprocal to divide by a fraction. In her efforts to understand the demands of teaching, she also taught mathematics to a sixth-grade class for a full school year.

http://www.math.uga.edu/~sybilla

Finally, Monk found the effects of a content-specific pedagogy course to be of the same magnitude as the effects of a content course.

Together, these three findings suggest that it is not just any content preparation that matters, but content relevant to classroom teaching.

"Diabolical" Experiments
(In the Interest of Science!)

- What would happen if we turned the heat up on "excellent" teachers, perhaps adding more and more students to their classes so we could study what begins to slip and when? By discovering the "limits to excellence," perhaps we could gain valuable insights into how to better prepare future teachers.

- What if we finesse teacher popularity by equalizing the "marginal product"? For instance, if we tell parents they can put their child in with the more popular teacher and allow more students into that teacher's class, does the marginal productivity of that teacher depreciate to become comparable to the less popular teacher with the smaller class.

- Does having an occasional poor teacher actually develop students' ability to be resilient in the face of less than ideal conditions and strengthen their ability to learn independently — enough so that you might want to have some "bad" teachers in the mix?

Adapted from a presentation given by David Monk.
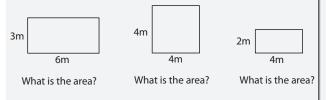
What We're Learning from Researchers of Teaching

In the past, the question about the mathematical preparation of teachers has turned on quantity: *How much* mathematics do teachers need to know? This led people to look at the number of mathematics courses taken. The assumption was that teachers simply need more of the standard offerings — classes typically designed with mathematics, engineering, business, and other students in mind. By looking at the mathematics that teachers need, however, we open the possibility that the mathematical content of courses for teachers might be different from existing offerings. By studying mathematical dimensions of the work teachers do, researchers have provided evidence that it is not simply a question of how much mathematics teachers need to know. Instead, the issue is one of understanding what mathematics plays a role in teaching and when and where it plays that role.

While thoughtful consideration of what it means to teach can be traced to ancient times, systematic research on teaching did not begin in earnest until the late 1900s. One explanation for this is the tendency to confound the study of teaching with the study of teachers. Studies of who goes into teaching tell us little about the underlying structure of the work. Likewise, a study of what teachers know is quite different from a study of the knowledge teachers need and use in their work. A second assumption is that a theory of *learning* directly implies a theory of *teaching*. Although theories of learning inform good teaching, they provide limited guidance for much of the work teachers do — managing groups, establishing classroom routines, dealing with parents, and so on.

Robust theories of teaching have yet to be realized. Researchers, however, have begun to identify fundamental characteristics of the work and to develop ways to represent the complex activity of teaching. (See, for example, Lampert, 2001.) In doing so, they offer insight into the type of mathematical knowledge needed for teaching and into how this knowledge translates to improved teaching and learning.

For instance, Ball and Bass use a "practice-based theory of mathematical knowledge for teaching" to analyze videotape of classroom teaching with an eye for the mathematical demands of the work (2003). They identify mathematical tasks teachers routinely do — judging the accuracy of definitions given in textbooks, choosing helpful examples, interpreting students' explanations. Lacking this knowledge and skill, teachers are likely to confuse their students and to be confused by their students. Research of this kind provides evidence that, at least on the face of it, what teachers know and can mathematically do plays a significant role in shaping instruction.

Mathematics for Teaching

Because many of her students confuse area and perimeter, Ms. Jovanovic designed a lesson to address their confusion. At the end of the lesson, she wants to give her students a single problem to assess their learning. Of the following, which would best serve her needs?

3m, 6m — What is the area?

4m, 4m — What is the area?

2m, 4m — What is the area?

Scott Baldridge, Lousiana State University, with a future teacher

Mathematics for Teaching

Ms. Madison wants to pick one example from the previous day's homework on simplifying radicals to review at the beginning of today's class. Which of the following radicals is best for setting up a discussion about different solution paths for simplifying radical expressions?

a) $\sqrt{54}$

b) $\sqrt{72}$

c) $\sqrt{120}$

d) $\sqrt{124}$

e) Each of them would work equally well.

Thomas Parker (Michigan State University) and **Scott Baldridge** (Louisiana State University) have developed mathematically rigorous courses for elementary teachers and have written a textbook to be used jointly with elementary school textbooks from Singapore. Problems and assignments in their textbook incorporate and analyze problems and presentations in the Singapore books.

http://www.math.msu.edu/~parker
http://www.math.lsu.edu/~sbaldrid

Designing courses— teaching teachers

Working in the trenches of teacher preparation, several mathematicians are diligently exploring new visions for mathematics courses for teachers. They seek to re-conceptualize the mathematics courses offered to teachers while honoring the legitimate concerns and needs of teachers.

Nicholas Branca, San Diego State University

Nicholas Branca (San Diego State University) develops courses that combine the teaching of mathematical content with the teaching of how to do mathematics. He aims to prepare teachers to hold productive conversations about mathematical ideas with their students. In support of this goal, he combines a focus on student thinking with a focus on mathematical explanation. He uses these two cornerstones to provide a foundation for helping teachers re-conceptualize their understanding of mathematics and develop the mathematical skills they will need on the job.

http://public.sdsu.edu/CRMSE/nbranca.html

With recent progress in understanding the mathematical work of teaching, education researchers have begun to develop tests for measuring this knowledge. Initial results suggest that these measures isolate a body of mathematical knowledge that matters for student learning.

In a longitudinal study of three major school-reform programs in 115 low-performing urban elementary schools, Hill, Rowan, and Ball (2005) investigated the effect of mathematical knowledge on student learning in first and third grade. Multiple-choice questions used in their measures ranged from ones about "common" mathematical knowledge expected of anyone, such as what number is halfway between 1.1 and 1.11, to "specialized" mathematical knowledge needed for teaching, such as those in the "Mathematics for Teaching" boxes (Ball, Thames, & Phelps, in preparation).

Mathematical knowledge for teaching, as measured by these tests, significantly predicted student learning. The table to the right gives estimated coefficients for a regression-like, linear model based on student variables, teacher/classroom variables, and school level variables (not shown). While the effect is small, it is meaningful in this setting. If a third-grade teacher performs one standard deviation above the mean on the multiple-choice measure, that teacher's students are predicted to gain 2.28 additional points on the student learning measure. Given an average gain of 39.3 point over the year, this is equivalent to 2 to 3 additional weeks of instruction per year. Looked at another way, the effect of teacher mathematical knowledge is comparable to the effect of socioeconomic status (SES) — historically the most robust predictor of student achievement.

Hierarchical Linear Model for Predicting Student Learning		
	Grade 1	Grade 3
Intercept	57.6	39.3
	(1.31)	(.97)
Student variables		
SES	3.96***	2.13**
	(.94)	(.76)
Teacher/classroom variables		
Methods & content courses	.53	1.64
	(1.00)	(.92)
Certified	.23	−.34
	(.89)	(.73)
Years experience	.72	1.04
	(1.14)	(.65)
Mathematical knowledge for teaching	2.22*	2.28**
	(.91)	(.75)
Lesson length	−.11	1.77*
	(1.04)	(.87)

  * Significant at $p < .05$
 ** Significant at $p < .01$
*** Significant at $p < .001$

(Standard errors given in parentheses.)

This result, if it holds up in further studies, has ground-breaking policy implications. Although the prospect of equalizing student socioeconomic status is remote, the prospect of improving teacher knowledge of mathematics for teaching suggests a relatively clear agenda.

To test the hypothesis further, Hill, Rowan, and Ball also ran the model including a similar measure of teacher knowledge for the teaching of reading. If teacher reading knowledge also showed an effect on student mathematical achievement or washed out the effect of mathematics knowledge, it would suggest that general intelligence or general knowledge of teaching were the determining factors. Instead, though, the effect of teacher knowledge of mathematics remained, and there was no effect of teacher knowledge of reading on the mathematics achievement of students.

These results suggest that the "right" content knowledge, that which is specifically connected to the work teachers do, matters for students' learning.

What We Can -- and Can't -- Learn from International Studies[3]

Student achievement results from international studies confirm suspicions that U.S. children aren't performing as well as we think they should. It might be tempting to conclude that differences in student achievement are due to the poor content knowledge of U.S. teachers. This conclusion is reinforced by additional sources, such as Liping Ma's proposed difference between U.S. teachers and Chinese teachers (1999), and stories that people tell from talking with teachers in the United States and in other countries.

The 1999 Third International Mathematics and Science Studies (TIMSS) provide the most complete international data we have with regard to teacher knowledge, teaching, and student achievement, but conclusions can be hard to draw.

Consider, for instance, the self-reported data about teacher preparation shown at right. There are no clear relationships between student achievement in a country and the reported major area of study of teachers. Compared to high-performing countries, the United States reports a somewhat lower percentage of teachers whose major area of study is in mathematics. One might conclude from this that more teachers should major in mathematics, but look at our neighbor to the north. Canada, which has a culture similar to ours, scored very well on this achievement test (in the top ten), but has a smaller percentage of teachers reporting mathematics as a major area of study. Relationships between teacher preparation and student achievement by country may not be straightforward.

Do differences in teaching help to explain variations in student achievement? In the 1999 study, videos of classroom lessons were gathered in the United States and in six countries that were high achieving in the 1995 study. Researchers examined over 75 different features of teaching that could be reliably coded. Although only a few features of teaching were different in the United States from the high-achieving countries, the variations may impact the learning opportunities available to students in the classroom.

One difference was the kind of problems given to students. Researchers classified problems into three types and examined their frequencies:

• *stating concepts* (recalling or applying definitions or conventions);
• *using procedures* (applying standard procedures); and,
• *making connections* (constructing relationships among facts, data, and procedures).

The third type of problem requires more than applying previously learned information. Examples include giving a mathematical justification, making a conjecture, or looking for a pattern.
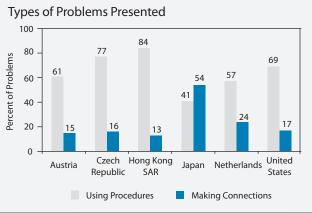
The *stating concepts* problem type was rare, but results for the other two types were more interesting. (See bar graph.) Hong Kong and Japan differed dramatically on these two types even though those were the two countries with the highest level of achievement in the sample. And, perhaps surprisingly, the United States does not differ much from many of its higher achieving peers.
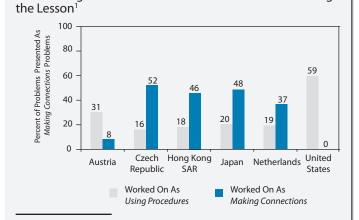
However, the percentages of how often these types of problems were presented to students may be a reflection of the curriculum being used, not of teaching or teacher knowledge.

3   Ideas in this section are adapted from a presentation given by James Hiebert.

TIMSS 1999 Achievement in Mathematics (Eighth Grade)
Teachers' Major Area of Study in BA, MA, or Certification
(Percentage of Students Whose Teachers Reported Each Area)

	COUNTRY	MATHEMATICS	MATHEMATICS EDUCATION	EDUCATION
HIGHEST PERFORMING COUNTRIES	Singapore	78	32	48
	Korea, Rep. of	56	61	19
	Chinese Tapei	82	39	32
	Hong Kong, SAR	57	30	36
	Japan	79	27	15
LOWEST PERFORMING COUNTRIES	Indonesia	67	49	29
	Chile	74	51	72
	Philippines	78	23	26
	Morocco	84	17	20
	South Africa	73	32	53
	United States	41	37	54
	Canada	22	19	49

Types of Problems Presented

Bar graph showing Percent of Problems (0–100) for Using Procedures and Making Connections by country: Austria 61/15, Czech Republic 77/16, Hong Kong SAR 84/13, Japan 41/54, Netherlands 57/24, United States 69/17.

How *Making Connections* Problems Are Worked On During the Lesson[1]

Worked On As *Using Procedures*	Worked On As *Making Connections*

[1] To analyze how *making connections* problems were worked on during lessons a fourth category was added, *giving answers*. Reported here are the percent converted to *using procedures* and the percent maintained as *making connections*.

Jim Lewis (University of Nebraska, Lincoln), chair of the Department of Mathematics and Statistics, has formed a partnership with Ruth Heaton, a colleague in the Department of Teaching, Learning, and Teacher Education. They teach and co-teach, in increasingly integrated ways, courses in mathematics and in methods for teaching mathematics. Working together, they deepen their students' understandings of mathematics while connecting that mathematics to the daily tasks faced by elementary school teachers. Along the way, they have observed, discussed, and improved their own teaching.

http://www.math.unl.edu/~wlewis1 *or* http://scimath.unl.edu/MIM

Surprised that the types of problems presented to students in the United States were similar to those presented in many of the high-performing countries, researchers wondered *how* the *making connections* problems get used during lessons. A second analysis following the *making connections* problems into lessons showed that teachers, intentionally or unintentionally, transformed problems when they implemented them in the classroom. A teacher might, for example, do part of the mathematical work for students, thus changing the nature of student experience, and potentially changing the problem from *making connections* to *using procedures*.

The graph to the left shows dramatic results. In this case, Hong Kong and Japan look almost identical. Although the two countries present a different percentage of *making connections* problems, such problems are handled similarly. In the United States, in contrast, these problems are simply not sustained. Virtually none of the *making connections* problems were implemented in a way that retained their expressed focus. That is, the apparent intention of the problem was changed while working it out, so students did not have the opportunity to wrestle with important mathematical issues.

This examination of both teaching and of the major areas of study for teachers suggests that, while it is risky to use large international studies to make claims about how teacher knowledge affects student achievement, relationships probably exist.

Additional support for the existence of these relationships comes from watching videos of teachers. Their teaching reveals different kinds of knowledge of mathematics. For example, as Czech or Swiss teachers develop explanations for their students, as Hong Kong teachers sequence tasks during a lesson, or as Japanese teachers recognize the potential of student contributions to develop an overall point in the lesson, all seem to use mathematical knowledge in significant ways. In particular, knowledge seems to play a role in sustaining the potential of more ambitious mathematical tasks. To the extent that teachers in other countries know more mathematics, this advantage appears to be wrapped very tightly with the specific kind of mathematics knowledge used in teaching. Unfortunately, though, it has not been possible in these international studies to document the kind of mathematical knowledge teachers need in order to provide particular kinds of learning opportunities for students.

We see from this last example that the question posed earlier about teacher knowledge changes a bit. Instead of asking whether mathematics is a major area of study, the question becomes: What knowledge do teachers need to implement *making connections* problems in ways that stay true to these problems? That's a much more precise question.

Ruth Heaton and Jim Lewis, University of Nebraska - Lincoln

What to Make of the Evidence

The conclusions from international achievement data are similar to the conclusions of economists and researchers about the specificity needed to document relationships between inputs and outputs in education. All three lines of work suggest that the mathematical knowledge of teachers is related to student achievement. Further, because the relationship happens through teaching, it is essential that the mathematical knowledge of teachers is usable in teaching.

While answers to problems of education might seem simple at first blush, the issue of empirical evidence can be tricky. Data are often subtle, conflicting, and hard to get. Results are often shaped by the way in which questions are asked and studied. Moreover, prudent decision-making requires interdisciplinary judgment.

What mathematics do teachers need to know?

In the same way that engineers need mathematics applicable to problems of design and construction and executives need mathematics useful in business applications, teachers need mathematics they can use in their work. The question, therefore, becomes one of identifying the mathematical knowledge, skill, and habits needed for teaching. This requires an understanding of the work of teaching, and in particular of the mathematical work of teaching.

Further, when identifying mathematical knowledge useful to teaching, it may be helpful, directly or indirectly, to *connect that mathematics to classroom teaching*. If we say teachers need to know the different meanings of subtraction, then it might be useful to identify some of the tasks where teachers are likely to use such knowledge: when sizing up the difficulty of word problems in a textbook, anticipating likely student thinking, or hearing the different interpretations children use to explain their thinking. This exercise can establish more definitively that teachers actually need this knowledge, can focus attention on what it is about this mathematics that is important, and can clarify when and how to use the knowledge.

Richard Askey, University of Wisconsin, Madison

Over the past two decades, researchers have begun to explore where, when, and how teachers use mathematics. While more study is needed, some useful domains have begun to emerge: mathematical practices, or doing mathematics, and mathematical tasks of teaching. Taken together, they begin to identify a body of knowledge for an *applied mathematics of teaching*.

How many sides does a circle have? (Joshie, Grade 2)

Example 1: The Doing of Mathematics

Although students certainly need to learn basic computation, success with mathematics also requires that students learn to *do* mathematics — to interpret and set up problems, to use notation and terminology correctly, and much more. It follows, then, that teachers, too, must be able to do mathematics.

Mathematics for Teaching

Use an area model of multiplication to show the product 2.4 x 1.5. Explain how you set it up, and how to interpret the answer from your picture.

Being able to do mathematics competently, however, is not enough for teaching. Because they are responsible for developing other people's performance, teachers must also recognize what it takes to carry out mathematical work and must have language to communicate to others about the process. Additionally, beyond knowing the mathematics and the mathematical processes they teach to students, teachers must perform certain mathematical tasks as part of the distinctive work they do.

Why do you just add a zero when you multiply by 10? (Maria, Grade 5)

One such mathematical skill is that of mathematical explanation. This is the bread and butter of mathematics teaching. Teachers present mathematical arguments and give mathematical justifications. Teachers explain new ideas and respond to the intellectual struggles of students. Children ask: Why, when you multiply by ten, do you add a zero? And why do you invert and multiply to divide fractions? Teachers must develop explanations that use the mathematical concepts children understand. Other professionals use mathematics, such as engineers, physicists or accountants. But they do not have to explain mathematics and mathematical procedures as regularly as teachers do.

Mathematics for Teaching

David calculated $\frac{8}{15}$ divided by $\frac{2}{3}$ as follows:

"8 divided by 2 is 4, and 15 divided by 3 is 5, so the answer is $\frac{4}{5}$." Is David's answer correct? Is David's method correct? Is this a generally applicable method?

Hung-Hsi Wu (University of California, Berkeley) has been giving three-week, all-day, mathematics-oriented summer institutes for elementary and middle-school teachers for over five years. Each institute is supplemented by five follow-up Saturday sessions during the school year. Through careful examination of the content of elementary school mathematics, Wu has come to appreciate subtle complexities in the mathematics in the elementary grades and has developed problems that expose shortfalls in student and teacher understanding.

http://math.berkeley.edu/~wu

Hung-Hsi Wu, University of California, Berkeley

A second mathematical skill needed by teachers is that of selecting representations wisely and using them effectively. When a teacher wants to show students, visually, that 2/5 is greater than 1/3, what are the options? What are advantages and disadvantages of using circles? A number line? What are techniques for drawing equal parts? It would be helpful if the teacher understood that the least common denominator, 15, could be used to compare these fractions, for instance using a 3 by 5 rectangle partitioned into an array of 15 pieces. Again, this goes beyond knowledge and skill typically learned in elementary school or required of other professionals.

Teachers also need to speak and write mathematics correctly and to recognize what it takes to do so. They need to invoke mathematical notation at appropriate times, in appropriate ways. Mathematical language has a function and logic unique to the discipline. For example, defining a term in mathematics is not the same as defining a term in literature or history. Defining "even numbers" is different from defining "irony" or "primary data source." And, in teaching mathematics, teachers need to know why definitions are important in mathematics. They need to know when a definition is called for, how to write a precise and usable definition, how to establish the equivalence among different definitions, and what the role of definitions is in mathematical arguments.

These mathematical skills constitute a central domain of mathematical knowledge for teaching. Teachers need to be taught what they are. But where in the current curriculum would teachers learn these things? Fortunately, a number of mathematicians and mathematics educators are redesigning mathematics courses for teachers to include this crucial knowledge.
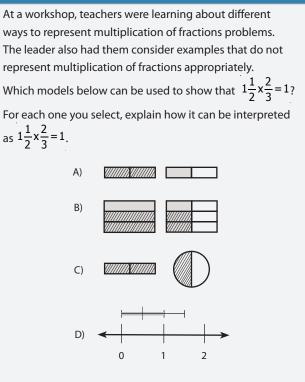
Mathematics for Teaching

For a lesson on comparing fractions, Ms. Banks wants to choose a model that will make it easy for her students to compare a wide range of fractions. What would be the advantages and disadvantages of using drawings of circles? Drawings of rectangles? Money?

How might the 3 by 5 rectangle below be used to compare $\frac{2}{5}$ and $\frac{1}{3}$?

Mathematics for Teaching

At a workshop, teachers were learning about different ways to represent multiplication of fractions problems. The leader also had them consider examples that do not represent multiplication of fractions appropriately.

Which models below can be used to show that $1\frac{1}{2} \times \frac{2}{3} = 1$?

For each one you select, explain how it can be interpreted as $1\frac{1}{2} \times \frac{2}{3} = 1$.

Example 2: Mathematical Tasks of Teaching

Another important domain of mathematical knowledge for teaching includes the mathematical tasks routinely demanded of teachers in their work — a kind of *mathematical problem solving* that teachers do.

For instance, while a class works on finding the intersection of two lines, a teacher might want an example where the coordinates of the intersection point are not integers, yet are "nice" fractions — ones familiar and easily managed by students. The problem of generating two lines whose intersection has non-integral coordinates is seldom posed in typical mathematics classes taken by teachers. The development of the skill to do so quickly, on the fly, is seldom the focus of instruction.

A course on mathematical knowledge for teaching might consider conditions under which intersection points have integral coordinates. Teachers would learn that the coordinates of the intersection point involve differences in the intercepts and differences in the slopes, and that a simple method for generating lines with convenient points of intersection is to pay attention to these differences. For example, working in slope-intercept form and choosing integral differences in the y-intercepts (-3) and in the slopes (8), where 3 and 8 are relatively prime, a teacher can quickly generate two lines:

$$y = 3x - 1$$
$$y = -5x + 2$$

In this case, the x-coordinate of the intersection is 3/8 and the y-coordinate is 1/8. The difference in slopes is the denominator for both coordinates, and because the two differences are relatively prime, the x-coordinate does not simplify, so students will have to contend with fractions.

Another example of a mathematical task of teaching is the sizing up of how a particular procedure is developed in a curriculum. For instance, a teacher might examine how the concept of "fraction" is developed across a fourth-grade textbook. Or, a teacher might compare different curricula, examining the various approaches to developing the subtraction algorithm.

Mathematical tasks abound in teaching. Teachers modify a problem to make it easier or harder while still addressing the same mathematics; they evaluate a student's unique approach to determine whether it would work; they choose examples with properties that will support an instructional point. Notice that these are all strictly mathematical tasks. These do not require knowing about students or about teaching. They are tasks that define the mathematical knowledge needed for teaching.

Roger Howe, Yale University

Mathematics for Teaching

Write a story problem that is modeled by the mathematical expression:

$$1 \frac{1}{4} \div \frac{1}{2}$$

Make a geometric representation of the expression and show how it represents each of part of the expression: $1 \frac{1}{4}$, $\frac{1}{2}$, and ÷.

Researching mathematical knowledge for teaching

As they have considered a role for themselves in the improvement of mathematics education, several mathematicians have become involved in research on the mathematical knowledge needed for teaching. Their efforts suggest ways for mathematicians to use their unique expertise to advance our collective understanding of the mathematics needed for teaching.

Roger Howe (Yale University) has been thinking about the curriculum — not just particular textbooks, but the issue of what to include. Given new expectations that all students learn algebra, how might arithmetic be presented so that it helps students learn algebra rather than poses a hindrance? He sees place value as an under-recognized leitmotif threading through a very large portion of the curriculum, affecting all of arithmetic, and connecting arithmetic to algebra. For instance, every integer (in fact, every decimal number) is the sum of multiples of very special numbers — powers of ten. These products of powers of ten are important enough that they deserve their own name, usable in elementary school (perhaps *place value numbers* or *very round numbers*). In this work he seeks to unpack and clarify mathematics in the elementary school curriculum that would forge connections and strengthen the curriculum.

http://www.math.yale.edu

Researching the Mathematical Knowledge Needed for Teaching

Hyman Bass (University of Michigan) worked with Deborah Ball, an education researcher, to investigate mathematical knowledge needed for teaching elementary school. Studying videotape, student work, and a teacher's notebook from an entire year in a third-grade classroom, he has learned to talk with teachers, mathematics educators, and policy makers about the resulting mathematical ideas. In his effort to understand elementary school teaching, he has apprenticed as an instructor for courses on teaching mathematics in elementary school.

Hyman Bass, University of Michigan

While these domains provide a start, we still need to better understand the mathematics teachers need to know and use in teaching. Here are three current lines of investigation.

Mathematics for Teaching

When asked to write *one thousand thirty four* as a number, many of Mrs. Carrera's students gave wrong answers. Of the following, which one are students least likely to write?

a) 134   c) 10304

b) 100034   d) 1000304

Researching Students

One effort to identify mathematics useful in teaching examines student thinking and the work teachers do to interpret and evaluate it. For example, in the Cognitively Guided Instruction project at the University of Wisconsin, researchers have examined student conceptions and misconceptions as a way to identify knowledge important for teaching. For instance, in learning to write numerals, young students who correctly write 47 may seem to regress as they learn about place value, writing 407 for 47 because they recognize that 47 is 40 and 7. As another example, because students come to see the equal sign as a signal to carry out the implied operation, they may answer 41 to the problem: $17 + \_\_ = 24$. While these specific issues require knowing something about students, they also require mathematical knowledge and skill. Mathematically speaking, why are these likely errors? What is the core issue of each? Researchers are using ideas such as these to design opportunities for teachers to learn this kind of knowledge.

Researching Teaching

A second effort to identify mathematics useful in teaching examines actual teaching. By watching classroom teaching with an eye for routine activities where mathematical knowledge would make a difference, researchers expose mathematical issues that matter for teaching. For example, Hyman Bass and colleagues at the University of Michigan have found that knowledge of mathematical definition arises more often and in more ways than one might at first suspect. Besides being terms to memorize, definitions have a place in classroom mathematical work similar to their place in mathematics research. A classroom discussion about whether the sum of two odd numbers is even or odd may break down because a teacher fails to recognize that students are reasoning from different definitions. Or, if communication becomes strained as students talk about a new idea, a label and a clear statement of what is meant might help. This work offers insights into when and where mathematical knowledge can be useful to teachers.

Researching Curriculum

A third effort to identify mathematics useful in teaching examines the textbooks and other materials teachers typically use. An analysis of these materials exposes mathematical knowledge and skill needed for using them well. It can also reveal mathematical demands of teaching that arise specifically in the work teachers do with textbooks — such as the mathematical knowledge and skill required to examine the development of a concept across chapters of text. For instance, Liping Ma has analyzed curricula for insight into the structure of increasingly difficult whole number addition and subtraction problems as shown at left. Such analyses begin to "unpack" the compressed knowledge adults hold of elementary mathematics.
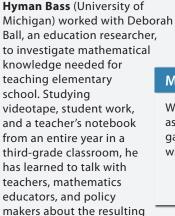
There is much yet to understand about the mathematical knowledge needed for teaching, but each of these lines of inquiry adds to what we know.

How number sense can be developed through arranged exercises

Within 100 (with regrouping)	45 + 18 = | 42 − 18 =
	38 + 25 = | 85 − 16 =
	27 + 4 = | 72 − 3 =
	52 + 12 = | 64 − 22 =
Within 100 (without regrouping)	63 + 20 = | 85 − 20 =
	63 + 3 = | 66 − 3 =
	40 + 5 = | 90 − 5 =
	30 + 20 = | 50 − 30 =
	11 + 6 = | 17 − 11 =
Within 20 (across 10)	15 + 2 = | 17 − 15 =
	6 + 9 = | 15 − 9 =
	8 + 4 = | 12 − 4 =
	6 + 6 = | 12 − 6 =
With 10	7 + 3 = | 10 − 3 =
	2 + 6 = | 7 − 5 =
Within 10	3 + 2 = | 4 − 1 =

Adapted from a presentation given by Liping Ma.

Would improved mathematical knowledge for teaching help reduce the "achievement gap"?

The uneven representation of different social groups in mathematics is a serious issue. And the unconscionable fact that student achievement in mathematics is predictable by language, race, class, or culture is a clear sign that the U. S. education system is not impartial. It is not enough to treat people the same. The issue is one of equity, where equity calls on the moral considerations of acknowledging differences and of dealing fairly with people in light of those differences.

Isn't equitable teaching, though, just a matter of "good teaching"? To answer that, it may be helpful to consider that "good teaching" might not always be equitable. A teacher might use sound pedagogical principles to design lessons and carry out instruction, yet lack important knowledge about differences among students or have small default patterns of acting that systematically advantage and disadvantage groups of students. For instance, a teacher might situate a mathematics problem in an everyday context to make it more meaningful to students, only to unwittingly assist some students and marginalize others. Of course, if "good teaching" is defined to include attention to equity, then, by definition, it is equitable. The point, however, is that equitable teaching may require considerations that go beyond the basic features of quality instructional design applicable in settings where achievement is not predictable by language, race, class, or culture.

Robert Moses and Ed Dubinsky, The Algebra Project

But what does the mathematical knowledge teachers possess have to do with equitable teaching? Equitable teaching requires *pronounced attention to tasks of teaching that work against systemic patterns of inequity in teaching and learning*. As such, it makes certain mathematical knowledge and skills indispensable. For instance, the task of hearing and interpreting student thinking becomes mathematically more demanding when that thinking and its expression are markedly different from one's own.

Teaching is fundamentally about linking course material to the experience of learners. Mathematics teachers must be able to move adeptly between students' experiences and the mathematical canon. They must establish a common experience for learners and channel this experience into productive mathematical activity. For example, developers of The Algebra Project (http//www.algebra.org) identify forms of talk that provide stepping stones from diverse student experience to disciplinary mathematics. They propose that teachers need to learn to translate among familiar "physical events," conversational "people talk," structured "feature talk," and symbolic mathematical expressions (Moses, 2001). Translating mathematics across these different forms is a mathematically demanding activity that requires specific knowledge and skill. Efforts such as the Algebra Project suggest that there is significant mathematical knowledge that is distinctive to equitable teaching.

Two examples of the mathematical work teachers do when attending to equity are *situating mathematics in contexts* and *making mathematics explicit*.

Example 1: Contextualizing Mathematics

To connect with students and to provide a "tool for thinking," teachers and textbooks often situate problems in a variety of real-world scenarios. Unfortunately, these contexts are rarely benign. Contexts open up a highway of possibilities — some desirable and some not. Contexts may support student thinking about new mathematical ideas or help them see the relevance of mathematics. However, those same stories and settings may raise unanticipated personal or social issues, distract students from the mathematics to be learned, or distort mathematical ideas and procedures. As teachers select and create contexts for situating mathematics, they must juggle consideration of varying student experiences with the mathematical integrity of the situations they use — a task requiring mathematical knowledge.

William McCallum, University of Arizona

William McCallum (University of Arizona) has been collaborating with teachers and mathematics education researchers to analyze the content of problems in algebraic thinking drawn from school curricula. These analyses, accompanied by sample student work, yield mathematical descriptions that support teachers in the classroom. Expanding his work on calculus reform, he has begun to address issues of mathematical knowledge needed for teaching at the university level and to explore the value of mathematicians' research work on education.

http://math.arizona.edu/~wmc

Nearly every context for mathematics problems serves to advantage some students and disadvantage others. This does not mean teachers should avoid contexts. Rather, it implies that they must be sensitive to pitfalls and make adjustments as needed. They also might level the playing field by familiarizing students with the context before beginning mathematical work. Teachers might use a context from the classroom in which everyone participated — permutations for lining up at the door for lunch, or comparing the ratios of girls to boys in different grades. In working with contexts, teachers need to respond to what students bring to the work without losing sight of the mathematics being taught. Mathematical knowledge becomes an important resource for the ongoing challenge of navigating around and climbing out of the pitfalls that arise.

Marta Civil, University of Arizona

The mathematics of a context can go wrong in many ways. Suppose a fourth-grade textbook asks students to compare, using fractions, the size of different states to the geographic area of the entire United States. Imagine then that a teacher with students born in many different countries allows her students to use states in other countries as well. This introduces an issue that did not exist in the original problem. Using only the United States means that larger fractions correspond to larger areas. Using multiple countries introduces multiple "wholes" and raises the issue of units. In making this change to the problem, it would be crucial for the teacher to know what it means mathematically to compare the fractional areas of states in different countries: how to express values with reference to their units, the potential need to convert between units, and different mathematical alternatives for handling

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Mathematics for Teaching

Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students focused on particular difficulties that they are having with adding columns of numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:
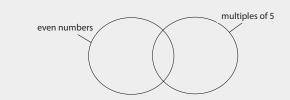
	A)		B)		C)
	¹		¹		¹
	38		45		32
	49		37		14
	+ 85		+ 29		+ 19
	142		101		64

Which two have the same kind of error?

likely confusions. The teacher must understand the ways in which Texas can be both larger than and smaller than the State of Baluchistan, Pakistan. Texas is larger in total area, yet it is a smaller part of its respective country — a situation that would not arise in comparing different states of the United States. Familiarity with mathematical notation, language, and content

is essential for managing the use of this problem in the classroom. Experience in identifying and addressing the mathematical issues that arise in making such changes would be helpful for teaching more equitably.

Or, imagine a second grade teacher asking students how many cookies each person in their family would get if their family shared 12 cookies equally. Besides being alert to ways in which the topic of "family" might bring into the classroom personal information about divorce, foster care, boyfriends, or siblings in prison, the teacher needs to help students move from imagining one's family sharing 12 cookies into the mathematics of dividing the number 12 by different numbers, say 2, 3, 4, 5, 6, or more. This involves a shift from the language of people, families, sharing, and cookies to the language of numbers, dividing, remainders, and fractions. What is the mathematical meaning of *remainder*, and how does the mathematical concept of remainder play out in the context of sharing cookies? Failure to address this question, and others like it, are at the root of systemic patterns of inequity. Likewise, in what ways is the sharing of cookies similar to and different from division of integers? In working on problems such as these, teachers need a firm grasp of what is being modeled, what mathematics is being brought to bear, and how to mediate between these two worlds. They need experience with mathematical modeling that considers the potential mismatches between situations and the mathematics that model them.

When we bring in everyday problems, we change the rules of the game. Students are often unsure what they should do. Are they meant to call on their everyday experience or are they meant to ignore it while they think about the mathematics being taught or tested? What tends to happen is that those who are most disadvantaged cannot read the unspoken rules. Many well-meaning teachers realize that there is a problem but lack the mathematical resources they need to act effectively and to teach equitably.

The work of the teacher is not only in creating the context but also in detaching the mathematics from the context, representing it in mathematical terms, then re-interpreting it back into the context. This is mathematical work, and the improvement of teacher knowledge and skill in this arena would help reduce current inequities.

Developing programs — developing mathematicians

In response to local school district needs, several mathematicians have launched programs designed to provide quality professional development in mathematics at sufficient scale to serve their communities, bringing together teachers, mathematicians, scientists, educators, representatives from the business community, school administrators, and others. Their efforts reinforce the sense that improving mathematical knowledge of teachers depends on respectful collaboration among mathematicians, educators, teachers, and the public.

William Vélez, University of Arizona; Carlos Cabana, San Lorenzo High School; and Ruth Cossey, Mills College

Example 2: Making Mathematics Explicit

Part of the mathematical work implied in the previous discussion is about being explicit with students about what is happening and what is expected. The call for teachers to be explicit first arose in research in multilingual settings, but it applies in any diverse setting and extends to nearly every aspect of doing mathematics.

Ken Gross, University of Vermont, with teachers

Ken Gross (University of Vermont) launched, in 1999, the Vermont Mathematics Initiative (VMI) — a statewide, three-year master's degree program designed to train elementary teachers to be mathematics leaders in their schools and districts. Bringing together mathematicians from higher education and master teachers from elementary schools, the program provides rigorous mathematics while also focusing on the use of that content in classroom instruction and school leadership. The program follows the adage that competence leads to confidence and seeks to produce graduates who view themselves as mathematicians, see the world in a mathematical light, and convey enthusiasm for mathematics to students and other teachers. To date, teachers trained by the program are in 40% of the elementary schools and 85% of all school districts in Vermont.

http://www.cems.uvm.edu/~gross

Student success in mathematics depends on the ability to carry out mathematical work effectively. Unfortunately, much of this activity is taught implicitly, and instruction is left to socialization. As a result, students who share the dominant culture have a huge advantage in reading the cues. While perhaps not altogether obvious, the more explicit a teacher can be about what is happening and what is expected, while still leaving intact the mathematical work to be done, the more likely it is that *all* students will understand what is meant.

Too often, communication about important aspects of doing mathematics relies only on indirect messages, such as marks and grades, or suggestive examples. Grades and examples fail to tell students what to do and why to do it. Needing to second guess what is expected, students from backgrounds different from the teacher are more likely to misunderstand these cues and less likely to learn.

For instance, in using drawings to reason about fractions, students need to be explicitly taught how to decide what kind of drawing to use for what problem as well as ways to label drawings to support their explanations. To understand the mathematical meaning of words such as *even* or *prime*, students need clarification about differences between the mathematical use of these words and their everyday use. (How are prime numbers similar and dissimilar to prime rib or prime time?) When solving a problem that requires manipulating an algebraic expression, students need to know how to decide when to simplify and when to expand an expression. In each of these cases, teachers need to teach students explicitly what to do, when to do it, and how to do it.

What does it mean to teach these aspects of doing mathematics? First of all, it means making them an explicit part of the curriculum. This was, at least in part, one of the motivations behind the National Council of Teachers of Mathematics (1989, 2000) process standards (reasoning, problem solving, communication, connections, and representations). This effort, and others, suffers from the fact that our understanding of the knowledge and skills that constitute mathematical work is inadequate and mostly tacit. A first step to more explicit teaching would be to annotate mathematical activity as it unfolds or to have students discuss their experience of what it takes to do mathematics. A second would be to identify for children a few key aspects of doing mathematics to make the focus of instruction, to choose problems and activities rich in these aspects, and to design instruction that explicitly addresses these aspects.

Another common case of inequity occurs when teachers tell students to "explain your answer" without providing explicit instruction on what this means. Directing students to "say how you did it" or "tell me why" leaves much unsaid. Giving examples, providing analogies, or referring to explanation as is taught in history or in science class, while perhaps helpful, is not enough. Mathematical explanation is a complex and subtle activity. Students need clear guidance about what will or will not count as an acceptable explanation. They need to be taught that logical steps are expected and that a drawing paired with an explanation is a propitious first step.

Imagine, for instance, in a mathematics content course for teachers, generating a list of criteria for mathematical explanation such as the one at right. This list is ad hoc and informal, but it offers suggestions of things to consider and ways to begin talking about explanation. It suggests a body of knowledge and skill that would support efforts to teach equitably.

While mathematical explanation does not lend itself to simple formulation, if teachers had tools for discussing explanation, they could be more explicit with students. For instance, identifying the steps in mathematical explanations and talking about when steps can be left out and when they must be included would help students who are not as able to pick up on subtle messages about acceptable and unacceptable steps. Short of this, indirect messages will continue to be more likely understood by students who share the teacher's background than by those who don't.

Recall, though, that the point here is that mathematical knowledge for teaching would do more than just lead to better instruction. It would also begin to address systemic inequities in business-as-usual teaching. Good teaching requires that teachers be able to explain and that they know what comprises mathematical work well enough to help novices learn it. However, to teach equitably, teachers must understand and be able to express what is involved in mathematical work well enough that they can be explicit with students about what is involved. What this means for mathematics teaching concerned with equity is that priorities shift so that selection and mediation of contexts, eclectic understanding of content, generous listening to the mathematical ideas of others, and explicit formulations of mathematical work come to the fore. How and where, though, do we create opportunities for prospective or practicing teachers to learn this skill?

Criteria for Mathematical Explanation
Generated by Prospective Teachers

1. Makes clear at the outset what is being explained, and why to start there, and carefully connects the explanation to the question or idea being explained
2. Starts from the beginning, and traces the logical flow of the reasoning
3. Is logical and complete, makes conclusion clear and links back to original question or claim or problem
4. Might number the steps if appropriate, or label parts of a diagram
5. Strives to be as simple and clear as possible
6. Defines terms as needed, uses available definitions as needed
7. Uses representations accurately (algebraic, geometric, etc.), and combining representations
8. Links the language and diagrams clearly to the steps of the argument
9. Shows what something means or why it is true, and is convincing to the person to whom you are explaining
10. Is calibrated to the context (considers the person to whom you are explaining, and what is already established as true and does not need more explanation)

Adapted from the 2004 Summer Institute sponsored by the Center for Proficiency in Teaching Mathematics, University of Michigan.

Mathematics for Teaching

Order these subtraction problems from easiest to hardest for students learning the standard subtraction algorithm, and explain the reasons for your ordering:

(a) 322
-115

(b) 302
-115

(c) 329
-115

Kristin Umland (University of New Mexico) and **Ted Stanford** (New Mexico State University) have launched teacher professional development programs funded through Mathematics and Science Partnership grants. They seek to increase teachers' confidence to tackle challenging problems and to help them learn to "think like mathematicians." Engaged with middle-school teachers, they consider what it would mean to "think like mathematicians" in the context of classroom instruction.

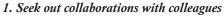http://www.math.unm.edu/~umland
http://www.math.nmsu.edu

Kristin Umland, University of New Mexico, with prospective teachers

What are mathematicians doing about the problem and what can you do?

Public education is a political arena, publicly financed, with many stakeholders. Teaching and learning are complex activities with social, moral, and cognitive dimensions. Where and how can mathematicians contribute their expertise?

The problem of mathematical knowledge for teaching is a good place to start. The sidebars of this report describe the work of a number of mathematicians who are engaged with this problem. Some are writing textbooks. Some are designing courses. Some are conducting research. And others are working directly with schools to design professional development for teachers.

Here are five additional ideas for getting involved:

1. Seek out collaborations with colleagues

We need collaboration between those with expertise in teaching and those who know the discipline of mathematics. Consult on a research project in education, review a K-8 mathematics textbook, or co-teach a course with an education colleague. For example, at Queens College of the City University of New York, mathematician Alan Sultan teaches a mathematics course for teachers while educator Alice Artzt offers a course on the teaching of mathematics (Sultan & Artzt, 2003). By observing each other's classes, and even asking questions in each other's class, these instructors have raised important issues. When collaborating, try to observe, listen, and ask questions. Express your views, though humbly. This helps to clarify issues and deepen understanding.

Risa Wolfson, Lawrence Hall of Science

Mathematicians from other institutions are another helpful resource. Sharing materials and approaches with others is an important step in building professional knowledge. You might contact mathematicians featured in the sidebars of this document, attend a Preparing Mathematicians to Educate Teachers (PMET) workshop (a project of the Mathematical Association of America, http://www.maa.org/PMET), or seek out connections with colleagues at conferences of the AMS, MAA, AMTE, or RUME.

2. Explicitly teach mathematical language, representation, and explanation

Modify courses, especially those for teachers, to include the use of mathematical explanations, representations, and language more explicitly. This change can be approached in small ways in parallel to existing course content. For example, at the start of a course, one might give students a problem that requires explanation or extensive work with mathematical representation. Then, after discussing student solutions, one could use responses to talk about choosing and using mathematical representations and giving mathematical explanations. Instructor and students can build a set of criteria like the example on page 18. Another productive activity is to have students listen to and interpret one another's explanations. Many students are reluctant to listen to their peers, but hearing out potentially confusing explanations is a central task of teaching.

Mathematics for Teaching

Mr. Fitzgerald has been teaching his class to compare decimals. He wants to pose a question that will show him whether his students understand how to put a series of decimals in order. Which of the following sets of decimal numbers will help him assess whether his pupils understand how to order decimals? Choose each that you think will be useful for his purpose and explain why.

a) .7   5   .09   3.2

b) .60   2.53   3.14   .45

c) .6   4.25   .565   2.5

d) These would work equally well for this purpose. They all require the students to read and interpret decimals.

Because these mathematical activities are so central to the discipline, the benefits of spending time on them can be far-reaching. At the same time, they are so familiar that they can be hard to see. As the anthropologist Clyde Kluckholm once said, "A fish would be the last creature to discover water."

3. Incorporate mathematical tasks of teaching

A third way for mathematicians to contribute is to experiment with mathematics problems situated in tasks of teaching. This might be done using artifacts of teaching, such as textbooks, student work, or video of K-8 classroom instruction. For instance, a problem that former students answered incorrectly might be recast as an incorrect solution for new students to analyze and explain what is wrong. Likewise, an instructor might have students decide on a set of examples that would exemplify important aspects of a topic they have just studied. After teaching a chapter on geometric transformations, give your students a list of problems and have them order them with regard to increasing conceptual difficulty and provide a rationale for their choice. In general, framing mathematical problems as tasks of teaching connects the mathematics to teaching in useful ways.

4. Learn more about K-8 mathematics teaching

The need to connect mathematics to teaching may, at times, feel overwhelming. After all, mathematicians are not trained to teach children and are unlikely to know the ins and outs of schools, classrooms, lesson plans, textbook adoptions, state standards, and so forth.

Teaching a mathematics course for K-8 teachers means you have to learn the mathematical content of the course. It does not mean you have to become a K-8 schoolteacher. That said, a little familiarity with teaching's central problems can help. Learn about common errors children make, about some of the shortcomings of textbooks, or about words and concepts that are difficult for children at particular grade levels. Spending time in and around classrooms, talking with a teacher, volunteering in an after-school tutoring program, or simply observing in a classroom may provide valuable insight that helps you connect the mathematics you teach to the work teachers do.

5. Support the generation and consideration of evidence

Current debate over the mathematical preparation of teachers occurs in a vacuum of disciplined evidence. Courses and programs are not designed from the synthesis of empirical work and little is done to produce evidence supporting or refuting assumptions on which they are built. We need to be mindful that trustworthy information costs money and requires involvement from different parties — mathematicians, education researchers, psychometricians, and educators — each contributing the professional expertise he or she has to offer and each respectful of the expertise others bring. Similarly, sensible use of information requires synthesis from an equally broad range of parties.

With these issues in mind, it is important to take time to discuss the construction and interpretation of evidence with colleagues from different backgrounds and different views. In particular, the design of studies deserves careful attention and knowledgeable review. In the end, decisions about education are democratic in nature and depend on our ability to talk with one another and to learn from our differences.

Dan Maki, Indiana University

Dan Maki (Indiana University) has been working with the College of Education and local school districts to develop effective professional development and to capitalize on relationships with teachers and schools to improve mathematics courses for prospective teachers. Among other innovations, he helped initiate one-credit courses for prospective secondary teachers, attached to specific undergraduate mathematics courses that link them to the content and pedagogy of secondary school mathematics.

http://php.indiana.edu/~makildui

Mathematics for Teaching

Mr. Chou noticed that many of his students gave wrong answers to the following problem:

$$8 + 3 = \underline{} + 5$$

Common wrong answers were 11 and 16. What might reasonably have led to these answers?

Looking Ahead

Many forces act against the changes proposed here — the size and scope of the problem, entrenched political stances, the institutional and philosophical divisions among mathematicians, mathematics educators, education researchers, and teachers, and our incomplete understanding of the body of mathematical knowledge entailed in teaching. What we do today, though, will shape the opportunities we create for the future.

Whose responsibility is it and who cares enough to do something? The improvement of mathematics courses for teachers, and of the mathematical preparation of teachers, should concern mathematics departments across the country for several reasons.

Mathematics for Teaching

Imagine you are a teacher working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
x 25	x 25	x 25
125	175	25
75	700	150
875	875	100
		600
		875

In each case, what method is being used? Could this method be used to multiply any two whole numbers? How can you be sure?

First, teachers prepare the students who arrive in university classes. The mathematics that prospective teachers learn, or do not learn, in courses taught by mathematicians plays an important role in determining what the K-8 students of those teachers learn.

Second, mathematics courses for prospective teachers help finance mathematics departments. Elementary school teachers represent one of the largest career groups in the United States, with approximately two million active K-8 teachers nationwide. And all of them take mathematics courses to become certified. Yet, like other fields such as business, engineering, and medicine, service courses for teachers need to serve. They need to be useful to professional work if they are to to remain viable. Losing these courses would be a huge financial loss, but it would also be a shame because mathematicians have valuable insight and perspective to offer classroom teachers.

Finally, teachers represent mathematics to children. In doing so, teachers lay the foundation for our society's understanding of mathematics and provide the most visible face of the discipline. If mathematicians care whether children learn mathematics, if they care about our country's economic and political vitality, if they care whether society holds them in high regard and funds them, then they should care about their largest public relations staff — teachers.

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