Critical Issues in Mathematics Education Series, Volume 3

Teaching Teachers Mathematics: Research, Ideas, Projects, Evaluation

Edited by Cathy Kessel

MSRI
Mathematical Sciences Research Institute

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The Humboldt Capstone Course
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Teaching Teachers Mathematics: Research, Ideas, Projects, Evaluation

Critical Issues in Mathematics Education Series, Volume 3

Editor: Cathy Kessel

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Preface

In 2004, the Mathematical Sciences Research Institute (MSRI) launched a workshop series, *Critical Issues in Mathematics Education*, to provide opportunities for mathematicians to work with experts from other communities on the improvement of the mathematics teaching and learning. In designing and hosting these conferences, MSRI seeks to encourage such cooperation and to lend support for interdisciplinary progress on critical issues in mathematics education.

The main goals of these workshops are to:

- Bring together people from different disciplines and from practice to investigate and work on fundamental problems of education.
- Engage mathematicians productively in problems of education.
- Contribute resources for tackling challenging problems in mathematics education.
- Shape a research and development agenda.

This booklet documents the fourth workshop in the series, *Teaching Teachers Mathematics*, held at MSRI on May 30–June 2, 2007. This workshop focused on mathematical preparation and professional development for teachers from kindergarten to grade 12. The nature, importance, and effect of mathematical knowledge for teachers was the topic of a 2005 MSRI workshop *Mathematical Knowledge for Teaching (K–8): Why, What, and How?* Discussions of how to assess such knowledge also occurred in the 2005 workshop, in the 2004 MSRI workshop *Assessing Mathematical Proficiency*, and in the 2007 workshop that is the subject of this report.

Much of the structure and content of this booklet comes from talks and comments by the workshop participants, especially Jeremy Kilpatrick, Raven McCrory, Hung-Hsi Wu, Judit Moschkovich, Cynthia Anhalt and Matt Ondrus, Ruth Heaton and Jim Lewis, Kristina Anthony, Susan Birnie, and Reuben Farley, Judi Laird, James Hiebert, Heather Hill, and Hyman Bass. Elaboration and augmentation of their remarks has come from the speakers' written work and project websites.

The workshop speakers were chosen for their ability to articulate widely-held perspectives on mathematics education, but this choice is not meant as an endorsement of those perspectives. The content of this booklet is not intended to represent the views of the organizing committee, the Mathematical Sciences Research Institute, or the sponsors of the workshop.

Acknowledgments

This booklet would not exist without all those who made the MSRI workshop come into being—the MSRI staff, the workshop organizers, the funders, the speakers, and the audience. This short list masks the expertise entailed. The workshop required the knowledge and skill of the MSRI staff and the workshop organizing committee. The speakers' talks and the audience responses reflected years of thought and commitment to mathematics education.

David Eisenbud, who was the MSRI director at the time of the workshop, suggested that this booklet be written. Jim Lewis and Deborah Ball provided guidance and feedback. The workshop speakers took time out of their busy lives to provide comments and corrections. Kathy O'Hara provided staff support while all of this was occurring. The booklet's design and layout are due to Liana King.

The MSRI workshop was made possible by funding from the National Science Foundation, Math for America, Texas Instruments, the Noyce Foundation, and the Center for Proficiency in Teaching Mathematics. Moreover, the National Science Foundation has funded many of the projects mentioned at the workshop and in this booklet.

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Introduction

This booklet discusses four aspects of the mathematical education of teachers in the United States:

- What is known about how teachers learn mathematics.
- Ideas about what teachers of mathematics need to know, based on research and professional judgment.
- Specific examples of how mathematicians and mathematics educators have been involved in the mathematical education of teachers.
- How efforts to teach teachers mathematics might be evaluated.

Much is unknown about how teachers learn mathematics in the United States. What is known suggests that often teachers do not have many opportunities to learn the mathematics that they need, either in teacher preparation programs or on the job.

But, there are encouraging examples of how mathematicians have worked to provide opportunities for teachers to learn mathematics. Some are projects with long histories. At the University of Chicago, I. N. Herstein began a mathematics program for Chicago public high school teachers and students, which was continued by Robert Fefferman. Although their focus has now shifted to earlier grades, Fefferman and his colleagues have continued to work with public school teachers. Among these colleagues is Paul Sally, who, in 1983, became the first director of the University of Chicago School Mathematics Project. This project has created a curriculum for students from pre-kindergarten through 12th grade, as well as professional development material for their teachers, and engaged in a wide range of other activities, including translation of textbooks and other educational literature. In 1991, Sally founded a teacher development program that leads to state endorsement for teaching of mathematics in the middle grades.

At the University of California, Los Angeles, Philip Curtis has been actively concerned about the mathematical preparation of incoming undergraduates since the 1970s. His concern led to the establishment of the Mathematics Diagnostic Testing Project for students in high school and college. In 1980, Curtis established the Visiting Mathematics High School Teacher Program which brings an outstanding high school mathematics teacher from the greater Los Angeles area to the mathematics department. These visiting teachers formed a network providing the mathematics department with access to the secondary mathematics education scene. In 1986, Curtis initiated the UCLA Teaching Internship Program in Mathematics, a joint program of the Mathematics Department and Graduate School of Education. This program provides financial support and additional mathematics education courses for mathematics majors planning to teach in high schools. In 2007, the UCLA Mathematics Department established The Philip Curtis Jr. Center for Mathematics and Teaching, which provides programs in teacher continuing education, K–12 student outreach, and undergraduate mathematics teacher preparation.

In 1947, Arnold Ross began a program at the University of Notre Dame for high school and two-year college teachers. A component added in 1957 grew into the well-known Ross Program for mathematically talented high school students which moved with Ross to Ohio State University, to the University of Chicago, and back to Ohio State. In 1989, several Ross alumni began the PROMYS Program at Boston University for high school students and extended it to high school teachers three years later. In 2003, ideas and experience from PROMYS became the foundation for Focus on Mathematics, a collaboration between Boston University, the Education Development Center, five Massachusetts school districts, and three colleges. Focus on Mathematics allows mathematicians to spend time in public schools and has created a new

graduate program for teachers at Boston University offered jointly by the School of Education and Department of Mathematics and Statistics.

Some projects do not have such long histories, but already have had statewide impact. For example, the Vermont Mathematics Initiative (VMI) began in 1999 as a collaboration between Ken Gross, a mathematician at the University of Vermont, and personnel at the Vermont Department of Education. VMI developed a three-year master's degree program for K–8 teachers at the University of Vermont. Now, VMI graduates represent over 90% of the school districts in Vermont, tests of students are showing positive results, and VMI has begun to address its next objective: Reaching all Vermont elementary and middle school teachers with a core set of mathematics courses, each taught by a mathematician and a VMI-trained teacher–leader.

In 1997, the UTeach Program for secondary teacher preparation started at the University of Texas at Austin as a collaboration between its colleges of education and natural sciences (including mathematics). Since UTeach's inception, the number of teachers graduating from the University of Texas at Austin has doubled. Moreover, UTeach has been successful in retaining students in its program and in their careers as teachers. Although within five years, almost half of new teachers leave teaching,[1] the corresponding retention rate for UTeach graduates is 70%. In 2007, the National Math and Science Initiative awarded grants to thirteen universities to replicate UTeach, including the University of California at Berkeley, the University of California at Irvine, and the University of Colorado at Boulder.

Among mathematicians, concern for the preparation of teachers and their knowledge of mathematics goes back at least to the time of Felix Klein's *Elementarmathematik vom höheren Standpunkte aus* (*Elementary Mathematics from an Advanced Standpoint*). Concern for school mathematics goes back to Euler's 1738 textbook *Einleitung zur Rechenkunst* (*Introduction to the Art of Reckoning*).

Educational research is a much younger field than mathematical research; and, correspondingly, concern for evaluating teacher knowledge and its effect on student learning occurred later among mathematicians. One example is the work of Ed Begle, which culminated in the manuscript for his book *Critical Variables in Mathematics Education.* After his death, the manuscript was edited by his students James Wilson and Jeremy Kilpatrick, and published in 1979 by the National Council of Teachers of Mathematics and the Mathematical Association of America. Begle's chapter on teachers concludes:

> Probably the most important generalization which can be drawn from this body of information is that many of our common beliefs about teachers are false, or at the very best rest on shaky foundations. . . . My overall reaction to the mass of information about teachers which is available to us is one of discouragement. These numerous studies have provided us no promising leads. We are no nearer any answers to questions about teacher effectiveness than our predecessors were some generations ago. (pp. 54–56)

Today, three decades later, the situation is less discouraging. There is still much that we do not know, but we do have some promising leads. It is fitting that this booklet begins with an account of the mathematical education of teachers based on Jeremy Kilpatrick's talk at the MSRI workshop.

1. R. Ingersoll, *Is There Really a Teacher Shortage?*, Center for the Study of Teaching and Policy, University of Washington, 2003, http://depts.washington.edu/ctpmail/PDFs/Shortage-RI-09-2003.pdf.

Based on a talk by Jeremy Kilpatrick

How Do We Teach Teachers Mathematics?

Jeremy Kilpatrick began his career in mathematics education as a junior high school teacher and is now Regents Professor of Mathematics Education at the University of Georgia. His research interests include mathematics curricula, research in mathematics education, and the history of both. He chaired the National Research Council committee that wrote *Adding It Up* and served on the RAND Mathematics Study Panel, which produced *Mathematical Proficiency for All Students*.

Teaching Mathematics to Teachers in the United States

It is difficult, if not impossible, to give a comprehensive account of how teachers are taught mathematics in the United States. This country has no single system for teaching mathematics to teachers. Although teachers are commonly prepared by taking undergraduate courses and completing a set amount of student teaching, the requirements for each vary by state and by undergraduate institution. Moreover, there are also alternative routes to teacher preparation such as teaching internships for career changers.

Opportunities for practicing teachers to learn mathematics also vary in significant ways. Some teachers must travel hundreds of miles to attend professional conferences or publisher's workshops for a few days. Others may have the opportunity to attend intensive summer institutes run by mathematicians or mathematics educators in their own community. Some teachers even have regular access to a mathematics specialist or teacher–leader in the same school.

Thus, we cannot assume much uniformity in the way teachers are taught mathematics during their preparatory coursework or during their careers. Instead of trying to consider *the* way in which teachers are taught (for there is no single system), one approach to thinking about how teachers are taught is to consider the following questions:

- Who teaches mathematics to teachers?
- What mathematics is taught?
- Who are the teachers?
- How should we teach mathematics to teachers?

Survey results, experience, and general impressions give partial answers to these questions and suggest some trends and commonalities in teachers' experiences. These answers may help those of us who teach mathematics to teachers to develop a more coherent vision of the mathematical education of teachers.

Who Teaches Mathematics to Teachers?

Teachers are taught mathematics by a variety of people. The standard route for teacher preparation involves undergraduate courses in two-year and four-year institutions. The mathematics courses in these programs are taught by mathematicians and mathematicians in training (i.e., graduate students) or by mathematics educators —people with degrees in mathematics education.

Practicing teachers are taught mathematics by members of the same groups—mathematicians and mathematics educators—and may also learn mathematics from the textbooks from which they teach, and from the accompanying teachers manuals. Unlike many preservice teachers, practicing teachers are also taught mathematics by a variety of people outside of academic institutions:

- Mathematics supervisors in schools, district offices, and state education departments.
- Commercial providers of professional development in mathematics.
- Teacher–leaders and coaches.

Finding out how many teachers are taught mathematics by people in these categories is extremely difficult, even for the case of teacher preparation. However, informal information and general impressions suggest that most of teacher preparation is done by faculty members in four-year undergraduate programs, usually in large public institutions.

Who are these faculty members? What is their preparation in mathematics education? It seems that more university mathematics departments are hiring people with degrees in mathematics education. This raises the question of how those departments determine how to judge their performance and determine criteria for promotion and tenure.

At the University of Chicago, Robert Fefferman and his colleagues work with Chicago public school teachers.

In general, it seems that universities such as San José State, Illinois State, or Georgia State that started out as teachers colleges or with a large teachers college component, mathematics departments have a relatively large collection of people who were trained as mathematicians, but have a strong commitment to teacher education. In contrast, "top 10" research institutions like Stanford or Berkeley tend to have a small collection of people involved in teacher education. Typically, they are a few people in the mathematics department—or else much of teacher preparation occurs in the school of education.

There seem to be few or no studies of how many U.S. teachers receive the first two years of their undergraduate mathematical courses in two-year colleges. However, estimates suggest that a large proportion do. For example, state sources in Maryland indicate that

40% to 60% of all Maryland teachers begin their preparation in community college.[2] A national report from the Education Commission of the States says, "Four out of 10 teachers have completed some of their math and science courses at community colleges." Such figures suggest that community colleges may play an important role in addressing teacher shortages as noted in reports from the National Commission on Teaching and America's Future and the National Research Council. The Education Commission of the States points out that some community colleges have begun offering baccalaureate degrees in education, allowing prospective teachers to bypass four-year institutions.

There seem to be increasing numbers of extended (e.g., five- or six-year) and alternative programs for prospective teachers. In 1998, 28% of those enrolled in teacher education programs began at the post-baccalaureate level—an increase as compared with previous years. Another trend seems to be that newer teachers are more likely to hold degrees in an academic field rather than in education.

Experience and tradition suggest that elementary teachers tend to major in education, but often take courses in mathematics departments. Many middle school teachers have at least a minor in mathematics.

This is consistent with survey findings. The 2000 National Survey of Science and Mathematics Education conducted by Horizon Research collected responses to questionnaires from 5,728 teachers in schools across the United States.

This information has several limitations. Like any survey, responses are self-report and rely on the participants' interpretations of the survey questions. Because state requirements for teacher preparation vary considerably, the findings are not representative of any particular state.

The results from the Horizon Survey give some indication of the number of undergraduate mathematics and mathematics education courses that practicing teachers have completed. The vast majority (91%) of K–4 teachers reported that they majored in education as did 72% of teachers of grades 5–8.

In-service Education in Mathematics in Past Three Years

Hours	K–4	5–8	9–12
None	14	14	7
Less than 6	22	15	8
6-15	32	29	17
16-35	18	19	25
More than 35	14	23	43

Source: 2000 Horizon Survey

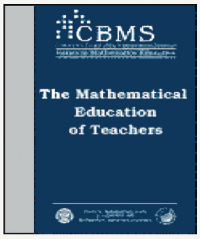
The survey responses indicate that practicing teachers do not often study mathematics. Most (86%) elementary teachers reported studying mathematics for less than 35 hours in three years—an average of less than 12 hours per year—as did the majority of middle and high school teachers (77% and 57%, respectively). If that is the case, then many teachers learn much, if not all, of the mathematics that they know before they start teaching.

Thus, this information suggests that, under the conditions that prevail in the U.S., preparation is very important, because it is the main opportunity that most teachers have to learn mathematics.

[2] M. K. DeBeal, "Rethinking the Community College Role in Teacher Training," *CBE Basic Education Online*, 2001, http://curie.umd.umich.edu/TeacherPrep/28.pdf.

What Mathematics Do We Teach to Teachers?

In 2001, the Conference Board of the Mathematical Sciences published *The Mathematical Education of Teachers*. This report, known as the *MET Report*, calls for "a rethinking of the mathematical education of prospective teachers within mathematics departments." Two of its major themes are the intellectual substance of school mathematics and the special nature of the mathematical knowledge needed for teaching.



The Conference Board of the Mathematical Sciences (CBMS) is an umbrella organization of seventeen professional societies, including the American Mathematical Society, the Mathematical Association of America, and the National Council of Teachers of Mathematics.

The *MET Report* focuses on teacher preparation and the design of mathematics courses for prospective teachers. Although the quality of such courses is more important than their quantity, measuring the quality of courses is difficult to do with a questionnaire. Consequently, large-scale surveys often focus on the number and names of courses involved in teacher preparation.

This is the case for the 2000 Horizon Survey which nevertheless offers interesting information about the education of teachers of mathematics. As might be expected, between 90% (9–12) and 95% (K–4) have taken a course in the general methods of teaching and most future teachers take at least one and often several courses that they describe as mathematics education courses. Interestingly, while 70% of high school teachers have had a course they classify as "supervised student teaching in mathematics," only about 40% of K–4 and 5–8 teachers report having had such a course.

Perhaps reflecting the generalist preparation of elementary and middle school teachers, teachers of grades K–8 are more likely to have taken courses in biology, physical science, or earth/space science than high school teachers, while high school teachers are more likely to have taken courses in physics, chemistry, and computer programming. Comparable percentages (ranging from 37% for K–4 teachers to 44% for 5–8 teachers) have taken a course in the instructional use of computers and other echnologies.

As to the number of mathematics courses that teachers of mathematics report having taken, the average varies from 3 (for K–4 teachers) to 4.5 (for 5–8 teachers) to 9.5 (for 9–12 teachers). Approximately 30% of K–4 teachers have had only one mathematics class and 50% have had two or less. Surprisingly, as many as 20% of 5–8 teachers have taken only one mathematics class and approximately 20% have taken only two.

Mathematics Education Courses That Mathematics Teachers Report Completing

	K–4	5–8	9–12
Number of semesters			
0	6	9	17
1	29	21	12
2	24	24	21
3 or more	41	46	50

Courses That Mathematics Teachers Report Completing

	K–4	5–8	9–12
Biology	77	71	49
Physical science	51	49	23
Earth/space science	41	42	20
Chemistry	31	40	47
Physics	19	26	52
Computer programming	12	29	63

Source: 2000 Horizon Survey

Courses Most Frequently Taken by K–4 and 5–8 Teachers

	K–4	5–8
Mathematics for elementary teachers	96	81
College algebra/trigonometry/ elementary functions	42	56
Probability and statistics	33	51
Geometry	32	37
Geometry for elementary/middle level teachers	21	28
Calculus	12	31
Mathematics for middle school teachers	5	28
Applications of mathematics/ problem solving	21	23

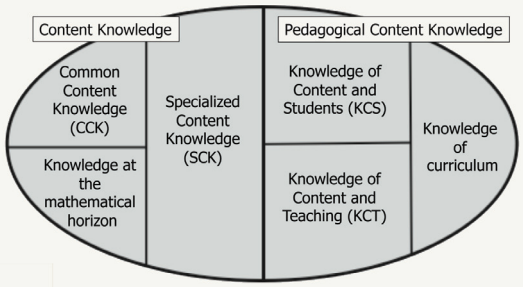
Source: 2000 Horizon Survey

No other mathematics course was taken by more than 20% of either group. These findings indicate that the course preparation of these teachers falls short of the recommendations of the *MET Report:* three courses (9 semester-hours) for K–4 teachers and seven courses for 5–8 teachers. The design and objectives of these mathematics courses for prospective teachers are described in the *MET Report,* which makes it clear that college algebra does not fall into this category.

Another way of thinking about teacher education is in terms of mathematical knowledge that is specific to teaching. This knowledge, often called "mathematical knowledge for teaching," has been characterized in various ways: as an application of mathematics to the practice of teaching or as the mathematics that is imperative—or useful, or important—for teachers to know.[3]

Characterizations of Mathematical Knowledge for Teaching

Domains of Mathematical Knowledge for Teaching



Mathematical knowledge for teaching, conceptualized as content knowledge (CK), not involving knowledge of students or teaching, and pedagogical content knowledge (PCK), an amalgam combining knowledge of content and pedagogy. CK includes common content knowledge used, as well, in other professional endeavors, specialized content knowledge, unique to teaching, and horizon knowledge that provides vision of content across its development. PCK includes combined knowledge of content and students, combined knowledge of content and teaching, and combined knowledge of content and curriculum.*

Learning Mathematics for Teaching Project

* See, e.g., D. L. Ball, M. H. Thames, & G. Phelps, "Content knowledge for teaching: What makes it special?," *Journal of Teacher Education,* 2008.

[Mathematical knowledge for teaching] allows teachers to assess their students' work, recognizing both the sources of student errors and their students' understanding of the mathematics being taught. They also can appreciate and nurture the creative suggestions of talented students. Additionally, these teachers see the links between different mathematical topics and make their students aware of them. Teachers with deep understanding are also more able to excite students about mathematics.

MET Report

"Knowledge of Mathematics for Teaching"

- Knowing (?) mathematics content
- Knowing which concepts are easy or difficult to learn and why
- Knowing ways of representing concepts so that others can understand them
- Knowing how to connect ideas to deepen them
- Recognizing what students might be thinking or understanding

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But the perspective that is too easy to miss and might be the most critical is:

– Experience thinking (and struggling) as a mathemetician does

Focus on Mathematics

[3] For further discussion with respect to elementary mathematics, see *Using Math to Teach Math*, 2006, Mathematical Sciences Research Institute, www.msri.org/calendar/attachments/ workshops/318/MSRI%20MKT%20booklet%20july28.pdf.

The *MET Report* takes the view that such knowledge is likely to develop only after years of professional study, but that its foundation must be laid during teacher preparation.[4] Consistent with this view, the Educational Testing Service comments, "The pre-employment phase consumes only about one-tenth of a full 40-year, lifetime teaching career. It is during the time spent teaching in the classroom that teachers, given proper support, evaluation, feedback, and targeted professional development, can evolve and mature into accomplished, exemplary teachers."[5]

We can also think of mathematics for teaching as the mathematics that teachers should learn. Just as the school mathematics curriculum is a selection from all that could be taught and all that students should know, so is the curriculum of mathematics for teaching. The intended outcomes of schooling reflect the values of our society. Research is not (and may never be) at a stage that indicates how these outcomes might be achieved. Thus the topics and construction of school curricula are not completely based on empirical research. [6]

Teacher preparation depends on the mathematics that teachers are to teach. In some cases, research shows that aspects of teacher preparation make a difference in teaching particular curricula or particular topics. The body of research is growing, allowing more decisions about teacher preparation to be based on empirical research rather than professional judgment.

What Do We Know About Teachers of Mathematics?

From the Horizon Survey, we know that teachers of mathematics are a fairly homogeneous group of people. Most are White and many have a master's degree. About half are over forty years old. The vast majority of elementary teachers are female, but almost half of high school teachers are male.

Many K–6 teachers are "self-contained," that is, they teach all academic subjects. The Horizon Survey found that all but 1% of these teachers consider themselves well or adequately qualified to teach mathematics. This is far more than the proportion of these teachers who consider themselves qualified to teach science.

The majority of middle school teachers report being very well qualified to teach computation. Many report being very qualified to teach geometry, a little more than half consider themselves very well qualified to teach algebra, but few report being well qualified to teach statistics.

The survey results for high school teachers indicate that most consider themselves very well-qualified to teach pre-algebra and algebra. These percentages decrease somewhat for geometry, and considerably for statistics and calculus.

4. See Recommendation 10 (p. 10) and the section on developing deep understanding (pp. 13–14).

5. *Where We Stand on Teacher Quality: An Issue Paper from ETS*, 2004, www.ets.org/Media/Education_Topics/pdf/teacherquality.pdf.

6. J. Hiebert, "Relationships between Research and the NCTM Standards," *Journal for Research in Mathematics Education*, 1999.

Some Characteristics of the Mathematics Teaching Force

Percent Teachers

Grade level	K–4	5–8	9–12
Female	96	76	55
White	90	86	91
Age			
41–50	31	27	29
51 +	27	30	30

Teachers' Reports of Their Qualifications to Teach Selected Subjects

Percent teachers

	Not well qualified	Adequately qualified	Very well qualified
Self-contained K–6 teachers			
Mathematics	1	39	60
Life science	10	61	29
Earth science	11	64	25
Physical science	21	61	18
Middle school teachers			
Computation	0	10	90
Pre-algebra	2	22	75
Algebra	11	40	49
Geometry and spatial sense	3	41	18
High school teachers			
Computation	1	11	88
Pre-algebra	1	5	94
Algebra	0	5	94
Geometry and spatial sense	4	26	70
Functions, pre-calculus concepts	6	34	61
Statistics	23	51	26
Calculus	39	36	24

Source: 2000 Horizon Survey

But how does a teacher's preparation affect the learning of his or her students? This relationship has been more studied for the case of high school mathematics than for any other part of the school curriculum. Most studies report that the more undergraduate mathematics that high school teachers have studied, the better the performance of their students. However, the effect is small and may decrease beyond five courses. It may be larger for advanced as opposed to remedial courses.[7]

After a meta-analysis of research published since 1990, the education researchers Robert Floden and Marco Meniketti remark, "*What* mathematics prospective teachers should study needs further examination." Their evidence allows them only to conclude, "Whether a degree in mathematics is better than a degree in mathematics education . . . remains disputable."[8]

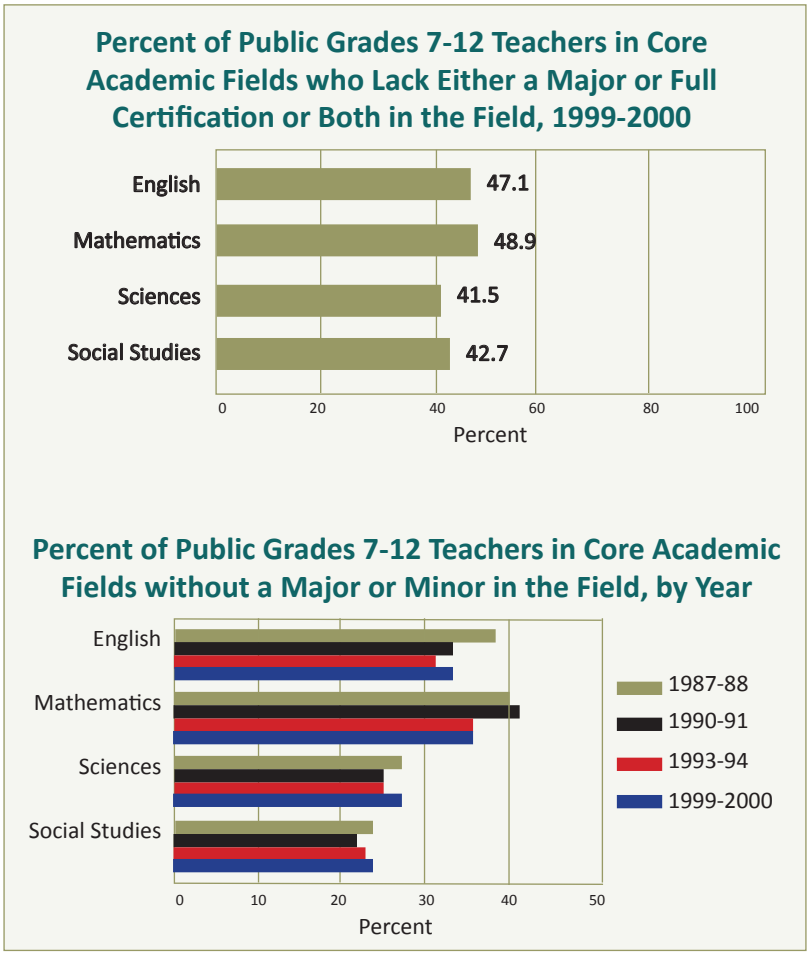
The evidence available suggests that certification is desirable for middle and high school teachers of mathematics. Suzanne Wilson and Peter Youngs surveyed research on this subject. Seven of eight studies found that, on average, students of teachers certified in mathematics performed better on achievement tests than students of uncertified teachers. (The remaining study examined certification more closely, finding that teachers with certification from private schools were an exception.) One study examined students' performance on different kinds of mathematics problems. Students of teachers with more specialized mathematics training performed better on "high-level" problems, but similarly on "low-level" problems.[9]

[7] R. E. Floden & M. Meniketti, p. 267, "Research on the Effects of Coursework in the Arts and Sciences and in the Foundations of Education" in M. Cochran-Smith & K. M. Zeichner (Eds.), *Studying Teacher Education: The Report of the AERA Panel on Research and Teacher Education*, Erlbaum, 2005. Available in part via books.google.com.

[8] R. E. Floden & M. Meniketti, p. 283.

However, Wilson and Youngs point out a major limitation of these studies. Certification requirements vary considerably by state, making it difficult to draw firm conclusions without better knowledge of particular certification requirements.

Whether or not certification makes a difference, out of field teaching—teachers teaching subjects for which they have not been prepared or certified—has continued for decades. Although the proportion for mathematics has decreased slightly in recent times, the situation has not changed much from that of fifty years ago.



Source: Ingersoll, *Out-of-Field Teaching and the Limits of Teacher Policy,* 2003.

[9] Wilson & Youngs, pp. 613, 634, "Research on Accountability Processes in Teacher Education," in M. Cochran-Smith & K. M. Zeichner (Eds.), *Studying Teacher Education*.

How Should We Teach Mathematics to Teachers?

Discussions of teaching mathematics emphasize different aspects: problem solving, connections between school and advanced mathematics, and discourse. These suggest different, but not incompatible, ways to think about teaching mathematics in general and teaching mathematics to teachers in particular. Each of these perspectives brings a particular aspect of mathematics into sharp focus.

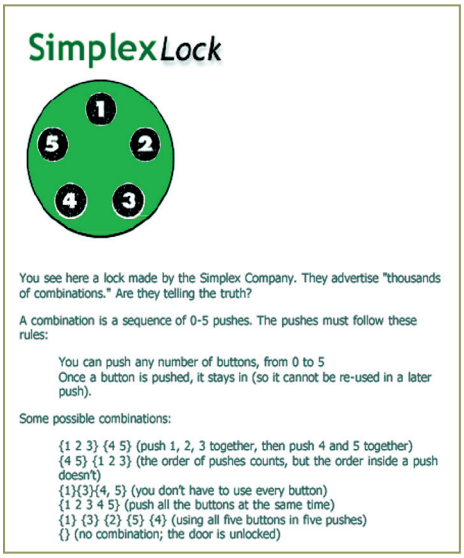
George Pólya

Problem solving. In his two-volume book *Mathematical Discovery*, George Pólya has sections on learning, teaching, and learning teaching.

In his discussion of learning teaching, Pólya distinguishes between *information* and *know-how*—the ability to use information, to do problems, find proofs, and other mathematical attributes. Teachers should have these attributes because "Nobody can give away what he [or she] has not got."

Similarly, the 2001 *MET Report* recommends that all courses for teachers help them develop the habits of mind of a mathematical thinker, along with mathematical knowledge. One approach to developing these habits of mind is to begin with a "mathematical immersion," as in the PROMYS program.

Problem given to teachers in PROMYS program



Source: www.focusonmath.org

Felix Klein

Connections with school mathematics. In *Elementary Mathematics from an Advanced Standpoint*, Felix Klein wrote, "For a long time prior to its appearance, university men were concerned exclusively with their sciences, without giving a thought to the schools, without even caring to establish a connection with school mathematics. What was the result of this practice? The young university student found himself . . . confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly."

Klein's book was intended to address this discontinuity between undergraduate mathematics and school mathematics. "My task will always be to show you the *mutual connection between problems in the various fields,* a thing which is not brought out sufficiently in the usual lecture course, and more especially to emphasize the relation of these problems to those of school mathematics." Rather than having prospective teachers quickly and thoroughly forget the great body of knowledge laid before them in their academic studies, Klein wanted them to draw from it a "living stimulus" for teaching.

Klein's concern for the "mutual connection" between undergraduate and school mathematics is echoed in the *MET Report*'s recommendation that core mathematics

courses be redesigned to help future teachers make "insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching."

Discourse. The historian and philosopher of science Jens Høyrup used extended episodes from history to analyze how the character of mathematical practice depended on its institutional setting. Teaching did more than transmit mathematical knowledge. It influenced mathematical practitioners' discourse—their values, their goals and methods of achieving those goals, and their identities. One important aspect of mathematical discourse concerns reasoning and proof. Methods of argument changed as mathematics developed over the centuries. Høyrup's *In Measure, Number, and Weight* notes:

> What was a good argument in the scientific environment of Euclid was no longer so to Hilbert; and what was nothing but heuristics to Archimedes became good and sufficient reasoning in the mathematics of infinitesimals of the seventeenth and eighteenth centuries—only to be relegated again to the status of heuristics in the mid-nineteenth century.

Although its surface features have changed, mathematics remains, in Høyrup's terms, a "reasoned discourse." Teachers need to learn to participate in this discourse—acquiring its essential goals and values as well as the ability to make reasoned arguments.

The Importance of Teaching Teachers

In modern mathematics, when a piece of mathematics is taught in a class, discussed in a seminar, or published, it enters the mathematical community and becomes a part of what we do. Classroom teaching, supervising graduate students, seminar talks, lectures, and publishing are all forms of teaching—and teaching preserves mathematics. Topics and techniques that are no longer taught, often drop out of sight, for example, solid geometry and taking cube roots by hand. Unless a topic, a technique, or other piece of mathematics is taught, there is an important sense in which it is not preserved. However, mathematics is more than topics and techniques. The practice of mathematics involves know-how as well as information —habits of mind and the predilection to justify arguments as well as the ability to do so. These also must be taught in order to preserve mathematics.

Those who teach mathematics are keeping it alive. Thus, those who teach teachers mathematics are keeping it alive for future generations.

Connecting Advanced Mathematics to High-school Content

In a course for future high school teachers at Rutgers University, students are asked:

Consider E = 0.9999
What should E mean?

Some students say E is 1 and others say it is controversial. (All of the students have passed a course in real variables.) This sets the stage for discussion of the Completeness Axiom.

Jens Høyrup

Proof and Justification in the *MET Report*

Mathematicians need to help prospective teachers develop an understanding of the role of proof in mathematics. . . . Prospective teachers at all levels need experience justifying conjectures with informal, but valid arguments if they are to make mathematical reasoning and proof a part of their teaching. Future high school teachers must develop a sound understanding of what it means to write a formal proof.

Based on a talk by Raven McCrory

Raven McCrory worked as a civil engineer before earning a Ph.D. in education. She is associate professor in the Department of Teacher Education in the College of Education at Michigan State University. Her primary area of interest is in mathematics education, with a focus on teacher knowledge and teacher learning. She is the principal investigator for the ME.ET Project.

What Do We Know About the Mathematical Preparation of Elementary Teachers?

Findings from the Mathematical Education of Elementary Teachers Project

Survey findings and experience suggest that under the conditions that prevail in the United States, the main opportunity that most teachers have to learn mathematics is while they prepare to teach rather than during their careers.[10] Because most teachers are prepared by taking undergraduate courses, mathematics courses for prospective teachers are a key leverage point for increasing teachers' mathematical knowledge.

In 2001, the *Mathematical Education of Teachers Report* made detailed recommendations for the preparation of all pre-college teachers of mathematics. In particular, it recommended that:

> Prospective elementary grade teachers should be required to take at least 9 semester-hours on fundamental ideas of elementary school mathematics.

Much of the systematically collected data about the mathematics courses that teachers actually take—and teacher education in general—comes from the Conference Board of the Mathematical Science surveys and from the 2000 Horizon Survey.

The CBMS 2005 Survey estimates that 72,000 students are enrolled in mathematics courses for elementary teachers in four-year institutions. At two-year colleges, there are approximately 29,000 students in mathematics courses for elementary teachers. Some of these students may become teachers without entering a program at a four-year institution; 30% of two-year programs offer all the mathematics classes required for certification.

How Many Mathematics Courses Do Elementary Teachers Take?

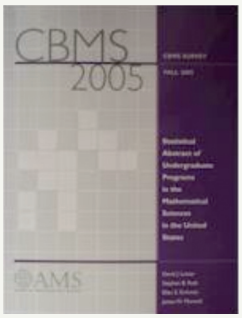
Results from the CBMS Survey suggest that times have changed: The average number of mathematics courses that states and four-year institutions require for prospective elementary teachers has increased, especially for teachers of later elementary grades. (Requirements may also have increased at two-year colleges in response to state requirements, but apparently these are not the subject of surveys.) Although the average has increased, course requirements at many mathematics departments still fall short of the 9 semester-hours recommended by the *MET Report*. For example, 16% of mathematics departments reported in 2005 that prospective teachers of later elementary grades were not required to take any mathematics courses.

[10] For further detail, see discussion of Horizon Survey findings on pp. 11–12.

Since the CBMS Survey in 2005, some states have increased the number of mathematics courses required or suggested for elementary teacher preparation. For example, July 2007 guidelines in Massachusetts now recommend 9 to 12 semester-hours of mathematics courses.[11]

Although the average number of mathematics courses required for elementary teachers has increased, we don't know much about what is taught in these courses nor do we know much about "what works" in these courses.

In 2006, the Mathematical Education of Elementary Teachers (ME.ET) project began a more in-depth examination of elementary teacher preparation. This project is designed to investigate the nature of the mathematics courses prospective elementary teachers are required to take as undergraduates in four-year institutions, the mathematics in these courses, what the teachers do learn and its effect on their teaching.



Every five years, the Conference Board of the Mathematical Sciences conducts a survey of undergraduate programs.

The project began by examining mathematics textbooks for preservice elementary teachers. Recently, there has been great interest in teacher knowledge. As the *MET Report* notes, "A number of mathematicians and mathematics education researchers have recognized the special nature of the mathematical knowledge needed for K–12 teaching and its implications for the mathematical preparation of teachers." Among mathematicians, one fairly recent outcome of this interest has been textbooks for elementary teachers. In 2003, three textbooks by mathematicians were published: Sybilla Beckmann's *Mathematics for Elementary Teachers*, Thomas Parker and Scott Baldridge's *Elementary Mathematics for Teachers*, and Gary Jensen's *Fundamentals of Arithmetic*. Around that time, two chapters of a textbook by Hung-Hsi Wu and a detailed account of a proposed series of courses for elementary teachers by R. James Milgram were posted on the Web.[12] These were among the twenty textbooks analyzed by the ME.ET project.[13] All were mathematics textbooks designed specifically for elementary teacher preparation. (Other textbooks used in mathematics courses for elementary teachers, like Van de Walle's *Elementary and Middle School Mathematics*, were categorized as methods textbooks for purposes of the textbook analysis.)

[11] July 2007 Massachusetts guidelines recommend 3 to 4 courses, http://www.doe.mass.edu/mtel/mathguidance.pdf.

[12] http://math.berkeley.edu/~wu/; ftp://math.stanford.edu/pub/papers/milgram/FIE-book.pdf.

[13] See R. McCrory, "Mathematicians and Mathematics Textbooks for Prospective Elementary Teachers," *Notices of the American Mathematical Society*, 2006, http://meet.educ.msu.edu/documents/McCroryNotices.pdf.

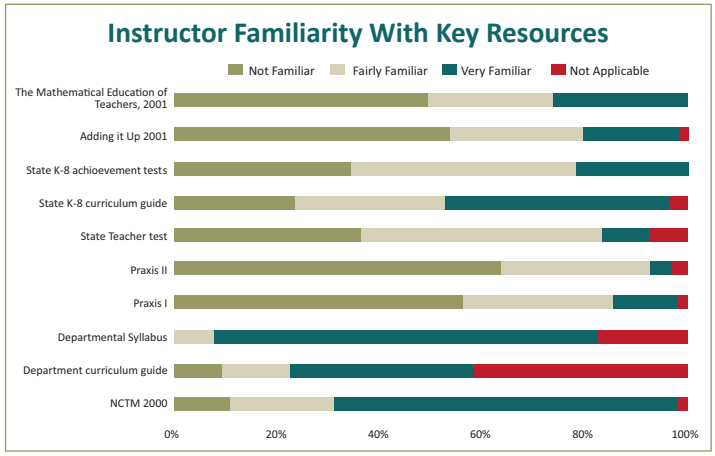
The ME.ET analysis found that mathematics textbooks written by mathematicians differed from others intended for elementary teachers in several ways: structure, coherence, rigor, and inclusion of claims about teaching. The books written by mathematicians tend to be shorter and more narrative, giving a sense of a mathematical landscape and delineating a course or sequence of courses. The opposite extreme was encyclopedic books that leave the instructor to choose among topics, problems, and activities.

Some of the textbooks went out of print after this analysis. In 2007, thirteen of the twenty remained. The next phase of the ME.ET Project obtained information about their use.

Mathematics Textbooks and Course Content in Elementary Teacher Preparation

In 2007 (its second year), the ME.ET project surveyed mathematics departments in three regions with different policy environments and student performance. These regions, South Carolina, Michigan, and New York City, have very different policies as regards teacher preparation. For example, South Carolina requires that teachers take the Praxis exam; Michigan and New York do not, but require that teachers take a state test.[14] All of the departments surveyed were at four-year institutions.

Student performance differs in the three regions. For grades 4 and 8, their average scores on the National Assessment of Educational Progress (NAEP) are similar to national averages. But in contrast to New York and Michigan, NAEP scores for black students are above the national average in South Carolina. In New York and Michigan, 36% and 38% (respectively) of all fourth-grade students are classified as "proficient and above" by NAEP, but 65% and 78% are deemed "proficient" by tests in their respective states. However, in South Carolina, similar percentages of fourth-grade students are classified as "proficient" by the state test and by the NAEP.



Source: ME.ET Project

The project was able to get survey responses from 55 instructors in mathematics departments in South Carolina and Michigan. (Instructors in New York did not participate.) Only about half of these instructors were familiar with the *MET Report, Adding It Up*, and the Praxis I and II (the certification exam offered by the Educational Testing Service which is required by many states). Such lack of familiarity with these reports and tests raises the concern that policy recommendations about teaching and the kind of knowledge that teachers need are not reaching these instructors.

[14] Details at: http://meet.educ.msu.edu/states.htm.

The ME.ET Project also interviewed department chairs or other department representatives about the content of courses required for elementary teachers. (The 57 interviews included departments in New York City as well as South Carolina and Michigan.) In over half (53%) of the departments, the primary focus of the first course was number and operation. The second course focused on geometry and measurement in 41% of the departments. In 26% of the departments, the second course focused on data and statistics.

In the departments surveyed, the most commonly used textbook was Billstein, Liebeskind, and Lott's *A Problem-Solving Approach to Mathematics for Elementary School Teachers* which was used in 12 departments. Next was Musser, Berger, and Peterson's *Mathematics for Elementary Teachers* (used in 7 departments). Almost all mathematics courses for elementary teachers had a required textbook, but surveys of instructors revealed that these books were used in varying ways—and not used at all by 5 of the 55 instructors.

Who Teaches Mathematics Courses for Elementary Teachers?

In the ME.ET survey of departments, all reported that mathematics courses for elementary teachers were taught in mathematics departments, with the exception of one institution. Most (59%) of these courses are taught by tenured and tenure-track faculty members. However, this varies by type of institution. The ME.ET Project found that at Ph.D.-granting institutions, 46% of these courses are taught by tenured or tenure-track faculty members. These proportions are quite different from findings for the national sample surveyed by CBMS in 2005 (see Table FY.1 in Chapter 5 of the survey report). For example, at Ph.D.-granting departments, 19% of the sections of mathematics courses for elementary teachers are taught by tenured and tenure-track faculty members.

Collaboration Between Mathematics and Education Departments: Michigan, New York City, South Carolina

% with moderate or great deal of collaboration

	BA	MA	PhD
Planning courses	50%	52%	55%
Designing cverall curriculum	36	36	36
Teaching together	21	8	0
Coordinating methods and content courses	21	50	20
Collaborating on research	21	28	30

Source: ME.ET Project

Most department chairs interviewed by ME.ET said that it was easy to find instructors and that the same instructors taught the courses for elementary teachers each year. Another question concerned collaboration with members of the education department. This is related to a *MET Report* recommendation that programs and courses for teachers be seen as a partnership between mathematics and mathematics education faculty. Chairs reported some collaboration between mathematics and education in various aspects of courses for elementary teachers. However, their responses may not reveal collaboration with mathematics educators in their own departments.

Do Prospective Elementary Teachers Learn from These Courses?

A Learning Mathematics for Teaching Item

Ms. Dominguez was working with a new textbook and she noticed that it gav more attntion to the number 0 than her od book. She ame across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

Which statement(s) should the sisters select as being true? (Mark YES, NO or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
a) 0 is an even number	77	71	49
b) 0 is not really a number. It is a placeholder in writing big numbers	51	49	23
c) The number 8 can be written as 008	41	42	20

Source: H. Hill, S. Schilling, & D. Ball, "Developing Measures of Teachers' Mathematics Knowledge for Teaching," *Elementary School Journal*, 2004.

For the ME.ET Project, students in teacher preparation courses in Michigan and South Carolina were administered Learning Mathematics for Teaching (LMT) items from the University of Michigan,[15] an attitudes and beliefs survey, and asked demographic questions (e.g., SAT or ACT scores, courses taken, mathematics major or minor). Useable data were obtained from 1704 students and their LMT scores were standardized to a mean of 50 points and a standard deviation of 10. There was a gain of 7.36 from pre-test to post-test. This is an effect size of 0.736 (where effect size is difference in means divided by the square root of the sum of the squared standard deviations). Because the items are scored using Item Response Theory (IRT) methods, it is hard to equate this gain with an increase in the number of correct items (harder items count for more in IRT scoring). It may be helpful, however, to think of it as a gain on the order of two items of "average" difficulty out of 25.

Items from the ME.ET Beliefs

i Math problems can be done correctly in only one way

j Mathematics involves remembering and application of definitions, formulas, mathematical facts and procedures

Survey

Strongly disagree	Disagree	Undecided	Agree	Strongly agree
1	1	3	4	5
1	2	3	4	5

Source: ME.ET Project

A regression analysis of these scores reveals that the students with the best scores on the pre-test were those who gained least from pre-test to post-test—a finding that made sense to most of the instructors interviewed. ACT and SAT scores were positively related to gain. Being a math major was also positively related to post-test gain. However, agreement with the beliefs survey item "Math problems can be done correctly in only one way" was negatively related to an increase in post-test scores.

This analysis only explains about 33% of the variance in scores and the project plans further analysis with respect to other variables. Updated analysis and technical information about the project are available at the ME.ET web site, http://meet.educ.msu.edu.

[15] http://sitemaker.umich.edu/lmt/home.

What Should We Teach Teachers About Mathematics?

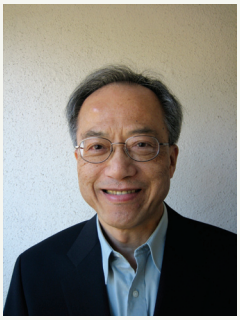
One Mathematician's Perspective

Current teacher preparation and professional development often do not address the basic characteristics of mathematics. These characteristics are hard to describe, but their absence is noticeable. Their presence may make mathematics easier to teach and easier to learn.

These basic characteristics are interdependent and may be described in different ways. Here is one possibility:

- **Precision.** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
- **Definitions.** Mathematical definitions are different from dictionary definitions because they are never circular. Whenever possible, mathematical definitions introduce new notions in terms of prior knowledge.
- **Reasoning.** From definitions, we can make deductions. These connect mathematics; they allow definitions and theorems to be connected by reasoning rather than given as separate, disassociated facts.
- **Coherence.** Mathematics is not a collection of separate unrelated notions. For example, the notion of natural number is extended by the notion of rational number in its fraction and decimal guises. Addition, subtraction, multiplication, and division on rational numbers extend the four operations on natural numbers.
- **Purposefulness.** A systematic exposition of a field of mathematics is parsimonious; it has no spare parts and no loose ends. Each topic occurs for a mathematical purpose.

There are two reasons why teachers should know this kind of mathematics. Teachers have a dual obligation: They must address the needs of the classroom and their students must learn good mathematics. However, teachers are unlikely to have good mathematics as their goal for student learning if they do not know good mathematics themselves. Moreover, understanding the basic characteristics of mathematics may allow them to be better teachers.



Hung-Hsi Wu has been concerned about mathematics education for two decades. He has taught summer courses and developed preparation courses for secondary teachers. He served on the committee that wrote *Adding It Up* and on the National Mathematics Advisory Panel.

Illustrating the Basic Characteristics of Mathematics

Precision and definitions. Use and misuse of the equals sign illustrate the importance of precision and the role of definitions. An important aspect of equality can be described in a very precise way by saying that equality is an equivalence relation. That is, the relation is:

Reflexive: A = A holds.

Symmetric: if A = B, then B = A.

Transitive: if A = B and B = C, then A = C.

However, it appears that in the United States, much more could be done with curriculum, teacher preparation, and professional development to help teachers and their students to learn the usual mathematical meaning of the equals sign.

The idea of equals sign as signaling "the answer" may be reinforced by usage found in some textbooks such as:

27 ÷ 4 = 6 remainder 3

Here, the equals sign is used as "an announcement of the result of an arithmetic operation." The left hand side of the expression denotes a number (when students have learned about fractions), but the right hand side does not. The information written above could be correctly stated as:

27 = (6 x 4) + 3.

Research on Students' Understanding of =

For at least three decades, researchers have reported on U.S. elementary and middle school students' understanding of the equals sign. Often, these students have an "operational" understanding of the equals sign—viewing it as indicating "adds up to," "produces," or "the answer follows."

For example, when asked "What does = mean?," children may respond:

"The equals sign gives the whole entire answer."

"It means like, if you have two and three, what adds up to."

"It's equals . . . . That's what it is. It means what it is, like three and two, what it is, five."

When asked, "Is 8 = 8 correct?," students may respond, "It doesn't make sense. It already tells you the answer."

When asked, "Is 13 = 7 + 6 correct?," students may respond that it should be the way around, i.e., that 13 should be on the right hand side of the equation.

Two explanations have been given for the prevalence of the operational view. One is that students are not developmentally ready to learn the meaning of equals as "the same as." The second is that elementary students are able to learn to learn this "relational" meaning of the equals sign, but that typical elementary school instruction does not support its development.*

* See Baroody & Ginsburg, "The Effects of Instruction on Children's Understanding of the 'Equals' Sign," *Elementary School Journal*, 1983; Seo & Ginsburg, "'You've Got to Carefully Read the Math Sentence . . .': Classroom Context and Children's Interpretations of the Equals Sign," in Baroody & Dowker (Eds.), *The Development of Arithmetic Concepts and Skills: Constructing Adaptive Expertise,* Lawrence Erlbaum Associates, 2003; Knuth, Stephens, McNeil, & Alibali, "Does Understanding the Equal Sign Matter? Evidence from Solving Equations," *Journal for Research in Mathematics Education*, 2006; Li, Ding, Capraro, & Capraro, "Sources of Differences in Children's Understandings of Mathematical Equality: Comparative Analysis of Teacher Guides and Student Texts in China and in the United States," *Cognition and Instruction,* 2008.

Reasoning and coherence. Students are told that:

- a fraction is a piece of pizza, part of a whole, a division, and a ratio;
- a decimal is a number obtained by counting hundreds, tens, ones, tenths, hundredths, thousandths, etc.;
- a percent is part of a hundred.

And, students are asked to use these to reason mathematically in solving problems.

Can these statements serve as a basis for reasoning? If a fraction is a piece of pizza, does that mean that two pieces of pizza can be multiplied? What is a "ratio"? If decimal and percent are as described, how does one compute with them?

The different meanings given to "fraction," "decimal," and "percent" raise questions about coherence. If fractions, decimals, and percents are all supposed to be numbers, why are they all different?

Some Questionable Definitions

1. Definition: Two non-vertical lines are parallel if and only if their slopes are equal.
2. Definition: Two non-vertical lines are perpendicular if and only the product of their slopes is –1.

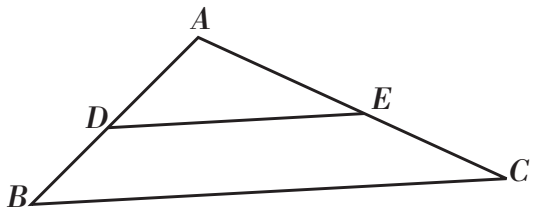
These "definitions" of parallel and perpendicular are not connected with geometrical definitions of parallel and perpendicular. One could argue that it's necessary to state (1) and (2) as facts for students, but not as separate definitions. However, a teacher needs to know that they are consequences of geometrical definitions and theorems, together with the definitions of the graph of an equation, linear equation, and slope.

Purposefulness. The history of the parallel postulate and Euclid's *Elements* illustrates mathematicians' concern for parsimony. The *Elements* is an exposition of the mathematics known in Euclid's time, showing how this knowledge could be derived from a small number of assumptions called postulates and common notions. Among these assumptions are five postulates about geometrical objects, one of which is the parallel postulate. A question that arose from this exposition was "Is the parallel postulate a 'spare part'—can it be proved from the other postulates?" This question was resolved in the nineteenth century by the development of non-Euclidean geometries, showing that the parallel postulate could not be proved from Euclid's first four postulates.

The *Elements* was organized without "loose ends." Each of its theorems could be proved from Euclid's postulates and common notions, either directly or from statements (lemmas and other theorems) that had been proved from the postulates and common notions. The exposition is arranged so that for each theorem, the necessary lemmas and theorems precede it in the text of the *Elements*. Conversely, each lemma and theorem has a purpose and is not a spare part; each is necessary for proving an important theorem or is important in its own right.

Similarly, in the mathematics of a well-designed curriculum, there are no spare parts or loose ends. "No spare parts" means that topics occur for a purpose, either as preparation for later topics or as topics that are important in themselves. "No loose ends" means that, when possible, the mathematical prerequisites of a topic occur in the curriculum before the topic is taught.

Such relationships among parts of school mathematics are not always made explicit in teacher preparation or in teachers manuals. A teacher who understands this will be aware that a topic to be taught is often mathematically dependent on prior knowledge, and look for mathematical relationships among the topics of a curriculum.



$AD / AB = AE / AC$

if and only if

$DE \mid\mid BC$

One example of how topics may depend mathematically on another comes from high school. The Fundamental Theorem of Similarity states that a line segment partitions two sides of a triangle into proportional segments if and only if the segment is parallel to the triangle's third side. This theorem plays an important role in high school mathematics.

A graph of an equation is often defined as the set of points that satisfy that equation.

Some consequences of this definition and the Fundamental Theorem of Similarity are:

- The graph of a linear equation $y = mx + b$ is a line.
- Two non-vertical lines are parallel if and only if their slopes are equal.
- Two non-vertical lines are perpendicular if and only the product of their slopes is –1.

Conclusion

Teachers need more than specific pieces of skills or concepts to improve students' achievement in mathematics. They need change in their perception of mathematics as a discipline that embodies its five basic characteristics.

Such a change cannot be accomplished in two- or three-day workshops. It requires sustained effort over a long period of time.

What Should We Teach Teachers About Mathematics and Language?

Findings from Research in Mathematics Education and Linguistics

The same word can have different meanings, depending on the social setting in which it is employed. For example, "discourse" has everyday meanings and technical meanings. Both the everyday meanings of "verbal exchange, conversation" and technical meanings might be used to describe speech in a classroom, while these meanings are related they are not identical. For academics engaged in discourse analysis, "discourse" can include spoken and written language—a lecture or piece of written text as well as a conversation. Moreover, it can include speech practices and values.

Judit Moschkovich's research has examined student understanding of algebraic and graphical representations of functions, conceptual change in mathematics, and mathematical discourse practices in and out of school.

Speech practices. Different communities may have different speech practices as illustrated by a study conducted by Shirley Brice Heath. She found that questions played different roles and were used in different ways in two rural communities. In the middle-class community, young children were frequently asked "known-answer" questions, in which the questioner already knew the answer. For example, when reading books with young children, parents pointed to pictures, asking "What's that?" In this way, children were learning to answer questions by giving information that they know is already possessed by their interrogator, a practice that is common in school settings.

Mathematicians, Teachers, and Discourse

The nature of discourse in the teacher community and the mathematician community is pretty different. In the mathematics community, we challenge each other very openly and freely. And teachers are often intimidated by mathematicians doing that. If we're in a group together, mathematicians just carry on, "What do you mean? How can you say that?" We challenge each other on the basis of, you know, whatever reasoning we think is going on in the paper. . . .

I heard from some teachers that it was nice for them to see the mathematicians challenging each other in ways that the teachers would interpret as attacks, and understand that wasn't what was going up. . . . So that was kind of a useful thing, I thought, in the process of the conference for people to start to understand the ways in which we talked in the different communities.

—MSRI Workshop Participant

In the working-class community, such known-answer rituals were far less frequent. Questions had different forms and signaled different activities. For example, "Did you see Maggie's dog yesterday?" signaled the beginning of a story. The appropriate answer was not "Yes" or "No," but another question, "No, what happened to Maggie's dog yesterday?"

At school, children from the two communities tended to respond differently when questioned by their teachers. The working-class children were puzzled by questioning routines that were familiar to their middle-class counterparts. However, their teachers learned about the types of questions that were familiar to the working-class children, and were able to gradually introduce known-answer questions in the classroom.[16]

[16] *How People Learn*, pp. 98–99. This can be read on the Web, see http://books.nap.edu/catalog.php?record_id=6160.

Social and national languages. Word meanings and questioning practices are just two examples of differences among communities. Word use, meanings of words, choice of words, language choice, speech practices, and topics of conversation can display the speaker's identity as a member of a particular discourse community and reflect that community's values. Characteristic practices in mathematics that reflect values such as precision and explicitness occur not only in mathematical writing, but in particular phrases and speech practices.

Speech Practices Valued in Academic Mathematical Communities

- Being precise and explicit.
- Stating an assumption explicitly: "let's say this is, "suppose."
- Making claims that apply to a specific set of situations.
- Connecting a verbal claim to a graph or symbolic expression.
- Conjecturing, arguing, proving.
- Use of phrases associated with proof: Special cases, extreme case, counterexample, existence proof.
- Comparing quantities.
- Abstracting, generalizing.

Different Word Meanings at Home and School

Everyday	School Mathematics
set the table	set
prime time prime rib	prime [number]
primo *(cousin, in Spanish)*	primo [number]
all numbers	any number
more/no more *(working-class UK households)*	more/less

Source: J. Moschkovich, "Examining Mathematical Discourse Practices," *For The Learning of Mathematics,* 2007.

Such phrases, words, and speech practices are part of what is called a *register* or *social language*, e.g., legalese or baby talk, which are ways of talking and meanings appropriate in particular social settings—law firms and parent–child communication. We can distinguish these *social* languages from *national* languages, such as English or Spanish.

In school, students' social languages and national languages may differ from the language of instruction in a variety of ways.

Some communities have national languages that are used exclusively at school. National languages from communities in which school mathematics has been taught are likely to have equivalents for terms of school mathematics. However, this is not universal. For example, in Tanzania, when the decision was made to use Swahili rather than English as the language of instruction, a Swahili equivalent for "diagonal" had to be created.[17]

A student's experiences may also influence his or her knowledge of the national language used in school and at home. In particular, bilingual students who begin their schooling in one country and then come to the United States may know some mathematical terms in English only. For example, a Latino/a student who completed high school in the United States may not know the Spanish word for "function."

[17] See, e.g., D. Pimm, *Speaking Mathematically: Communication in Mathematics Classrooms*, Taylor & Francis, 1989, pp. 93–94.

Bilingual mathematics learners. What does research tell us about bilingual mathematics learners? Older bilingual students are likely to carry out arithmetic computations in a preferred language, usually the language in which they learned arithmetic and may compute a miniscule fraction of a second more slowly if required to not use their preferred language. Bilinguals may also "code-switch," that is, they use two languages during one sentence or within one conversation. After the age of five, young bilinguals tend to speak as they are spoken to. If Spanish–English bilinguals are addressed in English, they reply in English; if they are addressed in Spanish, they reply in Spanish; and if they are addressed by a bilingual speaker they may code-switch in responding. Sometimes bilingual speakers will give an explanation in one language and switch to the second language to repeat the explanation. Researchers agree that code-switching is not random or a reflection of language deficiency—forgetting a word or not knowing a concept.[18]

Educating teachers. How can teachers become aware of issues like these and learn to address them? Here are some suggestions for working with teachers:[19]

- Provide opportunities, e.g., classroom cases, for teachers to examine English learners engaged in mathematical discussions. [20]
- Make room for assumptions and attitudes about language and language learners to surface.
- Focus mainly on students' mathematical thinking and reasoning rather than vocabulary or language proficiency.[21]

[18.] For details and references, see Moschkovich, "Using Two Languages When Learning Mathematics: How Can Research Help Us Understand Mathematics Learners Who use Two Languages?," National Council of Teachers of Mathematics Research Brief, March 2009, www.nctm.org/news/content.aspx?id=8468.

[19.] See also Moschkovich & Nelson-Barber, "What Mathematics Teachers Need to Know About Culture and Language" in *Culturally Responsive Mathematics Education*, Taylor & Francis, 2009.

[20.] See, e.g., elementary cases in J. Moschkovich, "Supporting the Participation of English Language Learners in Mathematical Discussions," *For the Learning of Mathematics*, 1999; secondary cases in Moschkovich, "How Language and Graphs Support Conversation in a Bilingual Mathematics Classroom," in *Multilingualism in Mathematics Classrooms: Global Perspectives*, Multilingual Matters Press, to appear.

[21.] See also J. Moschkovich, "Beyond Words to Mathematical Content: Assessing English Learners in the Mathematics Classroom" in *Assessing Mathematical Proficiency*, www.msri.org/communications/books/Book53/files/fm.pdf.

Collaborations and Coalitions

The ME.ET Survey (p. 21) found no instances of collaborations between mathematics and education in teaching courses for elementary teachers in research mathematics departments in Michigan, South Carolina, and New York City. However, these and other kinds of collaborations do exist—and have effects that extend far beyond one course, and can be considerably more effective than independent efforts from mathematicians, mathematics educators, and school districts.

Such collaborations are not entirely straightforward. Mathematicians and mathematics educators often have different views of mathematics, different ways of interacting with students in class, and different ways of assessing students. Collaboration may result in more awareness of differences, for example, in assessment—allowing the collaborators to be prepared to explain these differences to their students. Collaboration may also result in expanded views of mathematics or an expanded repertoire of teaching methods. These possibilities are illustrated in the two accounts of collaborations that follow.

A First Collaboration: A Mathematician Mathematics Educator Teach a Course at the University of Arizona

At the time of the MSRI workshop, Cynthia Anhalt and Matt Ondrus were post-docs at the University of Arizona. Anhalt's background is in mathematics education and Ondrus's background is in mathematics. They collaborated in teaching a semester-length course for 22 middle school teachers which was offered in conjunction with the Center for the Mathematics Education of Latinos/as.

The official goals of this course were to strengthen teachers' understanding of algebra and to discuss readings relevant to Latino students learning algebra. Both Anhalt and Ondrus found that they had unstated goals for the teachers—and that their unstated goals differed. Ondrus's goals concerned mathematics: be mathematically adventurous, patient, and self-reliant; do more than problem solving; understand what it means to be certain. Anhalt's goals also concerned mathematics, but were more closely related to teaching: reflect on everyday needs for teaching mathematics; deepen understanding of the mathematics they teach; gain new perspectives on issues of language, algebraic concepts, and their various representations.

They noticed other differences. In class, Ondrus tended to respond quickly, but Anhalt encouraged further explanation. Ondrus was struck by the way they planned for class. For example, Anhalt thought about how questions were asked. If they asked a question in class, how would it be phrased? Would it be asked of all or one student? Would students be asked to discuss it? Another difference was activities: Would they invent every activity they used in class or would they draw on existing activities? During their collaboration, some of these tendencies changed. Ondrus did not respond so quickly in class and considered using existing activities. Anhalt considered the possibility of inventing activities to use in class.

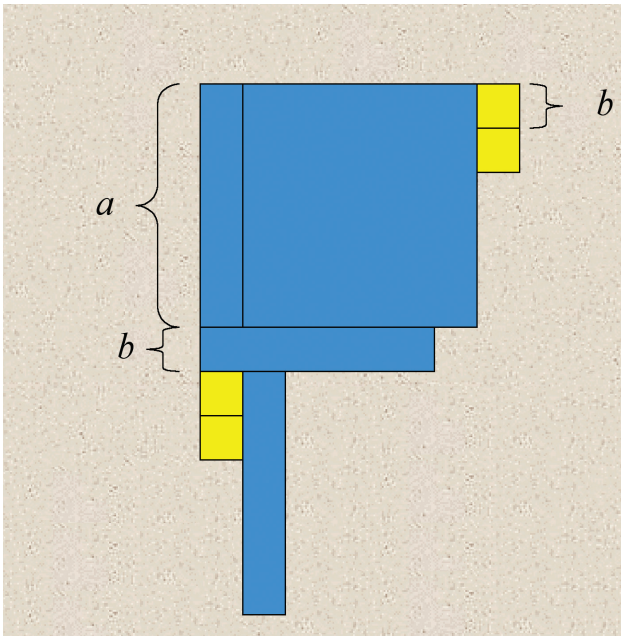
About a month into the course, Anhalt initiated the idea of an activity with algebra tiles.

Anhalt: Let's do perimeter with algebra tiles.

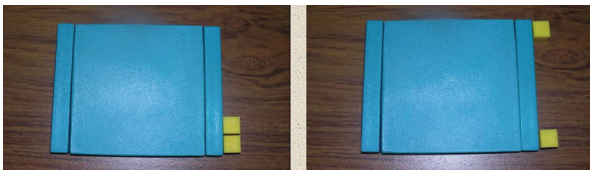
Ondrus: Huh?! Hmmm . . . here's what we can do.

Anhalt: What if we change . . . ?

What is the perimeter of this thing?



Eventually, the activity evolved into showing a shape using algebra tiles and asking the teachers to determine its perimeter.



$P = 3a + 8b + (a - 2b)$
$= 4a + 6b$

$P = 3a + 10b + (a - 2b)$
$= 4a + 8b$

The teachers were then asked to create similar problems.

This led to a discovery by the class:

Filling in a missing *corner* does not change the perimeter.

A teacher said, "Oh, it looks as if you fill in a missing chunk of one of these shapes then that doesn't change the perimeter." Ondrus had learned (with difficulty) to wait for the class to respond to incorrect statements rather than to respond himself. In this case, another teacher generated a counterexample. The incident led to a discussion of the importance of proving claims and using precise language.

Based on a talk by Jim Lewis and Ruth Heaton
For more information, see http://scimath.unl.edu/MIM/mim.html

From a University Program to a State-wide Program Collaborations at the University of Nebraska lead to Math in the Middle

Math in the Middle, a professional development project for the state of Nebraska funded by the National Science Foundation, began with a collaboration between a mathematician (Jim Lewis) and a mathematics educator (Ruth Heaton) at the University of Nebraska–Lincoln. In the late 1990s, two reports were produced that recommended such collaborations: *Educating Teachers of Science, Mathematics, and Technology New Practices for the New Millennium* and *The Mathematical Education of Teachers* (the *MET Report*). Jim Lewis was a co-chair of the committee that produced the first report and chair of the committee that produced the *MET Report*. A key recommendation of the latter is that the mathematics education of teachers should be based on partnerships between mathematicians, mathematics education faculty, and school mathematics teachers.

What are the advantages of such collaborations? Lewis points out that from a mathematician's perspective, there are several reasons to collaborate with a mathematics educator. He believes that the mathematical education of teachers is inherently an interdisciplinary process, thus it's appropriate that mathematicians and mathematics educators collaborate in teacher preparation. Moreover, one of the difficulties of teaching prospective teachers is their attitude toward mathematics—which often extends to the mathematicians who teach them. One advantage of collaboration with a mathematics educator is that he or she can support the mathematician's efforts.

Ruth Heaton, Lewis's collaborator, makes a similar point with regard to collaborations between mathematics educators and practicing teachers. Heaton began as a new faculty member at the University of Nebraska in 1995. During the next year, she began a partnership with the public schools and has now worked with the same school for over ten years. Because of this partnership, the prospective teachers that she taught were able to see connections between the methods courses that she taught and their field experiences in schools.

But these prospective teachers saw no connections between their mathematics courses and their field experiences. They were weak mathematically and had difficulty when mathematics came up in the methods course. And, they had little confidence in their ability to learn mathematics.

In the fall of 2000, Heaton and Lewis began working together.

Their collaboration began with Math Matters, a program designed to create a mathematics–education partnership, to link field experiences with the methods and mathematics courses, and to create mathematics courses for the prospective elementary teachers that allowed them to see the relevance of the mathematics in the course to school mathematics. From this beginning, Lewis and Heaton have subsequently developed a program that they refer to as "The Mathematics Semester" for prospective elementary school teachers. Students simultaneously enroll in a mathematics course taught by Lewis, a methods course taught by Heaton, a practicum experience in a local school, and another course taught by master teachers from the local school district.

Based on their four years of collaboration, Heaton and Lewis were well poised in 2004 to initiate, together with two collaborators, Heaton's department chair, Tom McGowan and Barbara Jacobson, Curriculum Director for the Lincoln Public Schools, Mathematics in the Middle (M^2), a program to improve teaching in the state of Nebraska. The first cohort of M^2 teachers earned master's degrees in 2006.

The Mathematics Semester

(For all Elementary Education majors)

MATH

- Math 300 – Number and Number Sense (3 cr)

PEDAGOGY

- TEAC 308 – Math Methods (3 cr)
- TEAC 351 – The Learner Centered Classroom (2 cr)

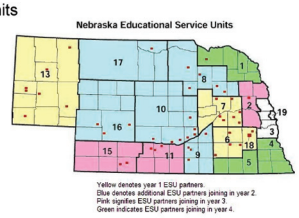
FIELD EXPERIENCE

- TEAC 297b – Professional Practicum (2 cr)
  - Students in elementary school classroom Mondays and Wednesdays
  - Math 300 & TEAC 308 taught as 3-hour block on Tuesdays and Thursdays
  - TEAC 351 taught by master teachers on Mondays
  - TEAC 297b supervised by master teachers

University of Nebraska Lincoln

M^2 Partnerships
People and Organizations

- 15 Educational Service Units
- 67 Local School Districts
- 91 Schools
- 130 Teachers
  - 27 teachers earned Masters Degree in 2006
  - 28 teachers will earn Masters Degree in 2007



Understanding Reactions to a Test

In Jim Lewis's mathematics course, students were given a test that included:

Is 250 a factor of 10030? Explain.

The students found the test difficult. Although they had learned about factoring, they had won a hard-fought battle to use calculators in class. When they used calculators, they got an answer that frustrated them and did not help them answer the question:

8.881784197 E44

Immediately after the test, the next class for the students was their methods course, taught in the same room by Ruth Heaton. At the beginning of class, a student raised her hand and asked whether Heaton thought the test was too hard. Rather than give an opinion or hold a discussion, Heaton asked the students to write about the test, letting them know that she would share their responses with Lewis. Three of the responses were:

> I believe this test, this class, this subject, are all difficult because they involve thinking in different ways than what we are used to. We have all been conditioned, in our own education; to believe that things are the way they are, and that's all there is to it. We haven't challenged ideas and proofs nearly as much as we should have. Asking "Why" to an idea or trying to understand the reasoning behind something is just not something most of us are used to doing. That's why this stuff is hard.
>
> I don't have a difficult time with abstract ideas. I love it when we work with new concepts. . . . I just want you to know that I have almost always been able to figure math problems out and I get VERY frustrated when I get stumped. I am very stubborn like that. Please don't take my temper personally.
>
> The major problem that I had was my reasoning for the factoring problem. I started off thinking that I should try dividing 250 into 10030, but the large numbers were daunting, so I panicked and tried using my calculator. The answer it gave me did not look pretty, which I think is what triggered my fall down a road of insanity (see my test for more details). Bad, bad calculators. . . . Once you started to explain the problem on the board, I wanted to smack myself in the head for being so silly.

Statewide Coalitions

Jim Lewis and Ruth Heaton's collaboration at the University of Nebraska led to a partnership involving the Nebraska public schools, which is poised to have a statewide impact on teachers —and, ultimately on student learning.

Such an impact has already been documented in Vermont, where a statewide program, the Vermont Mathematics Initiative, originated in 1999.[22] This, like Nebraska's Math in the Middle and the Virginia Mathematics and Science Coalition, is a coalition of K–12 schools and institutions of higher education, involving mathematicians, mathematics educators, teachers, and other school or district personnel.

Changing the preparation of all teachers of mathematics is an enormous task, but affecting all practicing teachers of mathematics seems even larger. Often, the attempt is made to reach individual teachers with professional development workshops. Teachers must then interpret and adapt the content of the workshops in their own classrooms.

Instead of focusing on workshops for practicing teachers, both the Vermont and Virginia coalitions prepare individual teachers to have a larger impact—on a school, district, or the entire state—rather than on a single classroom. This strategy has the advantage that professional development occurs in the context of teaching. Instead of occurring in workshops conducted by an outsider on "professional development days," professional development is done at the school by a specially prepared teacher. Current research suggests that this may be a more effective form of professional development. Teachers generally need professional development closely tied to their curriculum and the day-to-day work of teaching.[23]

[22] http://www.uvm.edu/~vmi/index_files/Page1434.html.

[23] Cohen & Hill, *Instructional Policy and Classroom Performance: The Mathematics Reform in California*, Consortium for Policy Research in Education, 1998; Gearhart et al., "Opportunities to Learn Fractions in Elementary Mathematics Classrooms," *Journal for Research in Mathematics Education*, 1999; Senk & Thompson, *Standards-Based School Mathematics Curricula: What are They? What do Students Learn?*, Erlbaum, 2003.

Based on a talk by Judi Laird. For more information, see http://www.uvm.edu/~vmi/

K–8 Professional Development in Vermont

In 1999, through the leadership of a visionary Commissioner of Education, Marc Hull, and Deputy Commissioner Marge Petit, the state of Vermont established a comprehensive, statewide, content focused mathematics professional development program, the Vermont Mathematics Initiative (VMI).

Evidence that such a program was needed came from several sources. In 1996, Vermont established high state standards for student achievement in mathematics, and—not unlike the rest of the nation—a substantial portion of the state's children could not meet the new high standards. Concurrently, new curricula were being adopted by school districts in Vermont that required of elementary teachers far more extensive mathematics content knowledge than the vast majority of teachers had received in their own college education. In 1997 and 1998, Vermont held Action Planning Institutes regionally across the state, and to nobody's surprise elementary teachers listed acquisition of the mathematics content knowledge necessary to teach to the new high standards as their number one need.

In response to these needs, in 1999 Hull and Petit sought the help of Kenneth Gross, a mathematician at the University of Vermont, to design a professional development program that would build mathematics knowledge and leadership capacity throughout the Vermont school system. Thus, the VMI was born as a three-year master's degree program at the University of Vermont. Originally designed for K–6 teachers, the VMI was seen to be equally effective with middle school teachers, and is currently structured as a K–8 program. At the heart of VMI is the idea of building a core of K–8 mathematics leaders who are deeply knowledgeable in mathematics and who can affect other teachers in their schools and districts.

Professional mathematicians in higher education and master elementary and middle school teachers have worked hand-in-hand in developing and implementing the VMI. The partnership of mathematicians and K–8 educators infuses all aspects of VMI, and allows the VMI to bring high-level mathematics and the classroom application of that mathematics together in a way that would be impossible otherwise.

Formal program evaluation, begun in 2004, has shown that the VMI has had a major impact on the teachers themselves, their classroom practice, and, most importantly, students in schools with VMI teachers. The evaluation studies reveal a consistent pattern of students in VMI schools outperforming those in control schools, as well as narrowing of the achievement gap between the free- or reduced-lunch eligible students in VMI schools and their non-eligible peers in matched schools.[24]

[24] http://www.uvm.edu/~vmi/index_files/2008%20VMI%20Evaluation.pdf.

Having trained sufficiently many teacher–leaders to penetrate deeply into the schools and districts of the state, in 2006 VMI introduced a second component, referred to as Phase II. This component of the VMI is designed to reach *all* K–8 teachers in a district with a core set of mathematics courses distilled from the master's degree curriculum in which the learning and transfer to the classroom is sustained through mentoring by the teacher–leaders.

By the summer of 2009, over 300 teachers had graduated with VMI master's degrees from the University of Vermont or were currently enrolled in the program. These teacher–leaders represent over 90% of the school districts in Vermont.

Strong mathematics content knowledge is the foundation of the VMI.

Phase I: The VMI Master's Degree. The master's degree program includes twelve mathematics courses, taught by mathematicians together with educators who for the most part are VMI graduates. The program starts with a deep understanding of arithmetic, which is foundational for the algebra, geometry, number theory, trigonometry, probability, statistics, and calculus that follow. The curriculum is designed to emphasize the connectedness of mathematics, and each course reinforces the learning that has taken place in preceding courses. Support structures are in place to help teachers master the content and to transfer mathematics content knowledge to classroom practice. The VMI signature course, Mathematics as a Second Language, lays the groundwork for the remaining courses.

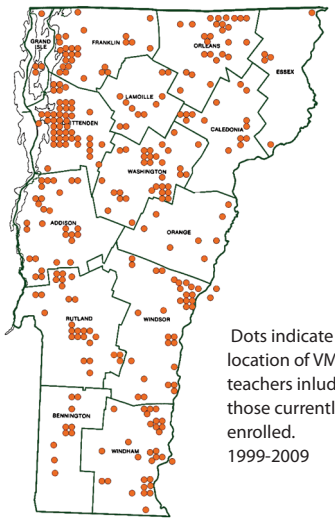
The VMI master's degree program is guided by four goals:

1. Strong and deep mathematics knowledge,
2. Effective mathematics instruction,
3. Action research that informs instructional decisions, and
4. Leadership that supports school-wide improvement of mathematics.

Goal 1, content knowledge, infuses all of the other goals, and is achieved primarily through the required course work. Goals 2 through 4 are closely interrelated and accomplished through

VMI Courses

- Mathematics as a Second Language
- Functions and Algebra
- Trigonometry for Teachers, and Algebra and Geometry II
- Measurement, Geometry, and Probability
- Number Theory
- Statistics, Action Research, and Inquiry into Effective Practice, I
- Statistics, Action Research, and Inquiry into Effective Practice, II
- Algebra and Geometry for Teachers, III
- Statistics, Action Research, and Inquiry into Effective Practice, III
- Calculus for Teachers, I
- Calculus for Teachers II
- Capstone VMI experience



Dots indicate the location of VMI teachers inluding those currently enrolled.
1999-2009

various means: classroom applications; regional workshops that focus on elements of effective teaching including formative assessment; mentoring by VMI master teachers who help connect the mathematics coursework to classroom practice; leadership training which includes the VMI Principal/Teacher Leadership Institute at which teachers and school administrators develop their "VMI Impact Plan" for improving mathematics instruction and learning in the school; and the three courses on Statistics, Action Research, and Inquiry into Effective Practice.

Phase II: District Implementation. This component of VMI is based on the course *Mathematics for the PreK–8 Educator,* an 80-hour six-credit experience. It is taught in local school districts and each district determines how the course is scheduled. Its content, developed by Kenneth Gross, is drawn primarily from the VMI master's degree courses *Mathematics as a Second Language and Functions and Algebra.* The course is co-taught by a mathematician and the district's VMI-trained teacher–leaders. Learning and transfer to the classroom is sustained and amplified through formative assessment, mentoring, and other meaningful experiences led by the teacher–leaders.

Leadership Roles Assumed by VMI Teachers

VMI graduates have assumed important mathematics leadership roles in their schools and districts. For example, they provide professional development to colleagues; many oversee their school's state-mandated testing and interpret its results; they serve on action planning and curriculum committees; and many serve as instructors in the Phase II VMI courses taught in their districts. VMI graduates also play significant leadership roles at the statewide level. For example, VMI-trained teachers make up an overwhelming majority of membership on statewide mathematics committees, including the committee that established the state's Grade Expectations and committees that have helped develop the assessments and expectations for No Child Left Behind. In addition, the VMI Executive Director (Judi Laird), Director of Phase II District Implementation (Robert Laird), and Lead VMI Instructor (Susan Ojala) are all graduates of the VMI.

National Impact

The VMI instructional model and course materials are now being used in several other states, including Illinois, Massachusetts, Nebraska, and New Mexico, and VMI has also been introduced in Newcastle, Australia. In addition to school districts in Vermont, the Little Rock (Arkansas) and Cincinnati (Ohio) Public Schools have adopted VMI for their mathematics professional development; and the Intel Foundation chose VMI to scale up as a national mathematics program. Notably, the Cincinnati program, called the Cincinnati-Based Vermont Mathematics Initiative, seeks to mirror to the maximum extent possible the comprehensive program that VMI implements in its partner school districts in Vermont.

Based on a talk by Kristina Anthony, Susan Birnie, and Reuben Farley. For more information, see http://www.vamsc.org/ and http://vamath.mspnet.org/index.cfm/

Preparing K–8 Mathematics Specialists in Virginia

In 1990, the Virginia Council of Teachers of Mathematics initiated efforts to place mathematics lead teachers in elementary schools. They were joined in this effort by a state-wide coalition, the Virginia Mathematics and Science Coalition, and by the Virginia Council for Mathematics Supervision. Soon, these groups agreed that lead teachers —regular teachers with additional duties—had more than enough responsibilities. The idea of lead teachers changed to the idea of mathematics specialists or coaches: Teachers without classroom assignments who could work at strengthening instruction for all teachers.

Recently, the notion of math specialist has received support from the state legislature. In 2005, the Virginia state legislature began work on a mathematics specialist endorsement and legislation mandating a math specialist for every 1,000 students in the state. In the spring of 2006, both houses of the Virginia Legislature passed a resolution commending Virginia school boards that employ mathematics specialists. In the fall of 2007, licensure regulations for math specialists in elementary and middle school went into effect.[25]

What is a Mathematics Specialist?

A team of mathematicians, math educators, and math supervisors worked very hard to create an agreed-upon definition for the role they called a "math specialist." This has been vital because otherwise school administrators and parents might simply view "math specialist" as an analogue of "reading specialist" and expect the activities and responsibilities of a math specialists to be similar, for example, pulling students out of class or responding to principals' requests.

The work of math specialists includes collaboration with individual teachers. They help teachers to use successful research-based instructional strategies, including those for students with limited English proficiency or disabilities.

> A teacher who has interest and special preparation in mathematics content, scientifically based research in the teaching and learning of mathematics, diagnostic and assessment methods, and leadership skills. The school-based mathematics specialist will serve as a resource in professional development, instructing children who have difficulties in learning mathematics, curriculum development and implementation, mentoring new teachers, and parent and community education.
>
> Proposed revisions to the Virginia Licensure Regulations for School Personnel

Who are Math Specialists?

Mathematics Specialists are teacher–leaders with strong preparation and background in mathematics content, instructional strategies, and school leadership. Based in elementary and middle schools, mathematics specialists are former classroom teachers who are responsible for supporting the professional growth of their colleagues and promoting enhanced mathematics instruction and student learning throughout their schools. They are responsible for strengthening classroom teachers' understanding of mathematics content and helping teachers develop more effective mathematics teaching practices that allow all students to reach high standards, as well as sharing research addressing how students learn mathematics.

University of Virginia
Math Specialist Project

[25] www.doe.virginia.gov/VDOE/Compliance/TeacherED/nulicvr.pdf.

I think of a math specialist not so much as a resource but as a mentor.

Grade 5 teacher, 2006

There are more math phobic teachers out there than you realize. I know because I never made it to the top of multiplication mountain myself. . . .

Last year, when I moved from grade 3 to grade 5, I was particularly scared about teaching a higher level of math. By supporting my mathematics instruction, [the mathematics specialist] has made me enthusiastic and confident.

Teacher, 2005

Having a mathematics specialist in my building has enabled me to gain valuable insight into how children process math concepts. . . .

More importantly, this knowledge has facilitated a process in which I can confidently provide instructional guidance that is necessary for teachers to effectively differentiate instruction in a manner that acknowledges the very best practices necessary to impact student achievement.

Elementary school principal, 2006

To do this, the math specialist might set up a study group for teachers. Or, the math specialist might teach classes with a classroom teacher for several weeks and debrief with the teacher after each lesson. After such work, it can be shared with teachers in grade-level meetings.

But a math specialist does more than work with teachers.

Math specialists also assist administrative and instructional staff in interpreting data, for example, results on state tests, and help the school to respond by designing approaches to improve student achievement and instruction. They ensure that the school curriculum is aligned with external standards.

Math specialists also assist administrative and instructional staff in interpreting data, for example, results on state tests, and help the school to respond by designing approaches to improve student achievement and instruction. They ensure that the school curriculum is aligned with external standards.

Finally, math specialists work with people outside the schools—parents, guardians, and community leaders. For example, a specialist might hold "Math Nights" or "Parents Nights," helping parents and guardians of students to understand what is going on in their child's classroom.

Preparation of Math Specialists

A Virginia math specialist must have completed at least three years of successful classroom teaching experience in which the teaching of mathematics was an important responsibility and have graduated from an approved master's level program, which includes at least 21 hours of coursework in undergraduate- or graduate-level mathematics. Over several years, the courses in this program have been developed, pilot-tested, and revised by university faculty in mathematics and mathematics education together with mathematics supervisors and master teachers from the school systems. The program is now offered at six universities: University of Virginia, Virginia Commonwealth University, George Mason University, Norfolk State University, Longwood University, and Virginia Tech.

Many of the courses are taught during four-week sessions at summer institutes. These summer courses are taught by teams with varied backgrounds. For example, an algebra course was taught by a university mathematics professor, a long-time mathematics supervisor, an advanced graduate student in mathematics education, and a secondary teacher. An educational leadership course was taught by two mathematics education faculty members from different universities and a secondary teacher. A number theory course was taught by a university mathematician, a second-grade teacher, and a fifth-grade teacher. A geometry course was taught by mathematician, a middle school teacher (now a math specialist) who taught in a rural area, and a middle school teacher who taught in an inner city public school. An advantage of having courses taught by such teams is that the course instructors have had a wide range of relevant experiences.

Other Degrees in Mathematics for Teachers

Aside from the degree program for math specialists, Virginia now has six master's degree programs for teachers in mathematics that involve various combinations of educational leadership. These programs are offered by the six universities which are a part of the Virginia Mathematics and Science Coalition. Because many of the courses in these programs were developed jointly for the math specialist degrees, these master's degree programs will allow between 15 and 21 credits for particular courses to be transferred between any of the collaborating institutions.

Courses for Math Specialists at the University of Virginia

Mathematics and Mathematics Education Courses

(18 credits)

- Numbers and Operations
- Geometry and Measurement
- Probability and Statistics
- Functions and Algebra
- Rational Numbers and Proportional Reasoning

Educational Foundation Courses

(9 credits)

- Curriculum: Advanced Theory (Mathematics)
- Development and Evaluation of Educational Staff
- Field Project (Practicum)

Loren Pitt, a research mathematician at the University of Virginia, has been a part of the Virginia Mathematics and Science Coalition since the 1990s. He writes of the coalition:

There are four essential lessons that I have learned from these experiences:

- Our effectiveness and impact was greatly magnified through collaboration;
- All constituencies in the education community brought essential knowledge and made essential contributions to the effort; and, the inclusive nature of the partnership contributed to making everyone a more valuable partner;
- Our partnerships grow stronger over time, but this will only be true when partnerships are built upon mutual respect and inclusiveness, and when the partnership's goals are unchanging and focused on real problems; and,
- A little luck and great partners are excellent assets.

Source: http://www.math.vcu.edu/g1/journal/Journal8/Part%20I/Pitt.html.

Based on a talk by James Hiebert

Measuring Effectiveness

James Hiebert is the Robert J. Barkley Professor in the School of Education at the University of Delaware. His research interests include classroom teaching and learning in mathematics, including international comparisons of teaching methods. He was the director of the Trends in Mathematics and Science 1999 Video Study of Mathematics Teaching and served on the National Research Council committee that wrote *Adding It Up*.

Why Measure Effectiveness?
Insights from Educational Research

At the MSRI workshop, there were many examples of how many mathematicians and mathematics educators are helping teachers to learn mathematics. Their efforts raise several questions:

- Are we making a difference? How would we know? How might we find out?
- If we meet again in five years, could we be sure that the recommendations we share with each other for helping teachers learn mathematics work better than those we are sharing today?
- In general, how can we get better at improving our efforts to educate teachers?

From some educational research perspectives, an obvious answer to "How would we know that we are making a difference?" is "Measurement." Collect evidence to show that teachers learn what we intend from our educational efforts. But, some might wonder whether measurement is needed at all. Doesn't it suffice to analyze the materials we use with teachers or the explanations we give during presentations? If these are mathematically sound, shouldn't our educational program be effective?

In brief, the answer is "Not necessarily, as shown by research in cognitive science." Cognitive scientists study how humans process information of all kinds—speeches, textbooks, teacher educators' explanations, classroom activities. Human minds are very active and process information in ways that make sense to them based on what they already know. The results can be surprising. For example, even when shown the correct arithmetic algorithms, young students can generalize procedures such as borrowing from 0 precisely, consistently—and incorrectly.[26] Much additional research illustrates the principle that people in general—not just children in classrooms—can interpret instruction and other forms of information in unexpected ways.[27]

Thus, it is unrealistic to expect that teachers will learn exactly what we intend, even if our mathematical presentations are beautifully crafted with impeccable logic. Assessment of teachers' learning needs to include careful collection and interpretation of empirical evidence.

High quality evidence can be used to measure the effects of our work with teachers, answering specific questions about whether or not the educational interventions made a difference. Did the teachers become better problem solvers? Did their beliefs about mathematics change? Did they learn the intended topics of geometry, or algebra, or fractions, with appropriate levels of understanding?

The type of evidence needed to answer the question depends on the kind of question asked. Questions might be about teachers' content knowledge of particular topics or about their ability to teach that content to students. The questions might ask about the effects of a one-week workshop or a year-long intervention.

[26] See, for example, Maurer's chapter in *Cognitive Science and Mathematics Education*, A. Schoenfeld (Ed.), Erlbaum, 1987.

[27] See National Research Council, *How People Learn*, National Academy Press, 1999, pp. 56-66, 121-124.

Whatever the questions, they need to be posed with enough clarity and precision to allow empirical answers. To collect evidence for a claim that teachers have become better problem solvers, have changed their beliefs, or have learned certain topics of geometry, one needs to use measuring instruments that actually show better problem solving, changed beliefs, or knowledge of particular topics.

Gathering high quality evidence means using tasks or instruments that are aligned with the questions and probe precisely the knowledge or behavior or beliefs that the questions ask about. Doing this might be as simple as giving prospective teachers two different versions of an off-the-shelf test before and after a course or professional development experience. Or it might be as complicated as videotaping the classroom teaching of graduates from a teacher preparation program and then analyzing the way in which their teaching reflects the aims of the program. No matter what form the evidence takes, it needs to be collected in a very systematic and careful way.

What counts as evidence of teachers' learning? This, of course, depends on what type of question is asked—what type of learning is of interest. However, it is useful to be aware that certain types of observations tend to have severe limitations. Anecdotes, for example, are especially problematic. Teacher educators often remember particular events from their interactions with teachers—Ms. Flores had a remarkable insight about the density of rational numbers during today's session or Mr. Jackson showed a really deep understanding of the fundamental theorem of arithmetic—and jump to the conclusion that the intervention was effective. These anecdotes, of course, are likely to be true for the particular case while saying little about overall effectiveness of the intervention. Similarly, perceptions—whether of the instructor or even the teachers—often have their own limitations. Perceptions of a changed attitude or new understanding do not necessarily indicate that this is indeed the case. There simply is no substitute for collecting direct evidence from all participants about the changes asked about in the questions guiding the intervention.

To conclude this discussion, it should be noted that the reason for going to all this work of measuring whether our education efforts have the effects we expect is more important than just helping each of us improve our own efforts. Ultimately, the major benefit of this work is that it allows us to share what we learn about effective interventions with each other. If we come together at meetings like the MSRI workshops on mathematics education and can only share opinions about what works well, we risk making very little actual progress in getting better at what we do. However, if we can describe educational efforts that show real effects, with trustworthy evidence, then others can borrow these ideas, try them out in different contexts, collect more evidence, and gradually adjust the intervention to yield increasingly good results.

Measurement, applied in this way, would preserve and pass along the best ideas we have developed for helping teachers learn mathematics. We could avoid what John Dewey called the saddest thing about American education—the fact that the best teachers take their best ideas with them when they leave. Each teacher must start over, inventing ways to help students learn. Likewise, teacher educators often must start over, inventing ways to help teachers learn. The antidote to this sad condition is for us to take seriously the challenge of studying our own teacher education efforts, to be clear about what we want teachers to learn and what questions we want to answer about the effects of our interventions, to collect evidence to answer exactly these questions, and to share what we learn with our colleagues. In other words, we need to see ourselves as part of a *profession* of people who teach teachers mathematics, as part of a *profession* of teacher educators.

Based on a talk by Heather Hill

Heather Hill is an associate professor at the Harvard Graduate School of Education. Her primary work focuses on developing measures of mathematical knowledge for teaching, and using these measures to evaluate public policies and programs intended to improve teachers' understanding of this mathematics.

Strategies for Measuring Effectiveness

Research Design Considerations

If you are teaching a course for teachers, you may want to know what effect the course had on those teachers. For course instructors, the standard way to measure that effect is by giving an exam at the end of the course. Perhaps you give the exam and all the teachers perform wonderfully well. Can you conclude their performance is due to your course?

Not necessarily. For example, the teachers might have been able to perform just as well on the exam before they took your course.

The next time you teach the course, you change your research design by giving the teachers the same test at the beginning and end of the course. (This might be described as a pre–post design.) The teachers perform poorly the first time and perform well at the end of the course. But, this design also has some flaws. Perhaps the teachers learned from taking the test twice. Maybe the teachers were enrolled in another course in the same semester they took yours, and that's the real cause of better performance. Were the teachers required to observe elementary school mathematics classes while they took your course? Maybe that's why they did so well on the second exam. Was there a "selection effect"—something special about the group of teachers that enrolled in your course? Maybe there was "contamination"; perhaps they aced your test because the exam was in the files of the local sorority or fraternity.

Each of these possible explanations is a threat to the validity of the claim that "the teachers' test scores improved due to their participation in my course." These and other threats to validity need to be addressed by design or considered as explanations for the improved test scores.

There are standard ways to think about designing educational measures and standard strategies for measuring the impact of educational experiences such as a course, a preparatory program, or professional development. Three important pieces are:

- Research question
- Research design
- Instrument or measure.

Research question. A standard question for evaluation is: What would individuals, for example, teachers in a course, have known and been able to do without this "treatment"? The treatment effect is an estimate of the difference between actual performance and this "counterfactual"—what would have happened if the same people had not been in the course.

In asking this question, care must be taken in identifying the treatment that is to be studied.

For example, is the question about the effect of:

- A textbook?
- A course?
- Course and instructor?
- A sequence of courses and instructors?
- A program?

Studying the effects of different treatments may require different research designs. For example, studying the effect of a textbook is quite different from studying the effect of that textbook as used by a given instructor. The effect of a textbook needs to be separated from the effect of any one instructor that uses it, requiring a large sample of instructors using the same textbook—and adding administrative burden to the study.

With respect to logistics, the easiest of these treatments to study is the effect of a course with a given instructor. The research question then becomes "What is the effect of course X as taught by instructor Y?" Depending on the effect of interest, the question can be further sharpened. "What is the effect of course X as taught by instructor Y on Z?" where Z might be teachers' problem solving abilities, their beliefs about mathematics, or their knowledge of elementary geometry.

Some Quasi-experimental Designs

Post-test only

Treatment group: Posttest

Threats: "Natural" learning over time, selection of treatment group, no comparison group, no pretest.

Post-test with comparison

Treatment group: Posttest

Comparison group: Posttest

Threats: "Natural" learning over time, learning from instrument (the test), selection of treatment and comparison groups.

Pre–post design

Treatment group: Pretest, posttest

Comparison group: Pretest, posttest

Threats: Learning from instrument (the test), selection.

Longitudinal design

Treatment group: Pretest, test, test

Comparison group: Pretest, test, test

Threats: Selection.

Advantage: Allows for growth modeling.

Research design. It might seem that an answer to this question requires evidence that is collected according to an experimental design in which teachers are randomly assigned to the treatment (the course as taught by a particular instructor). While ideal, this is often not possible. However, "quasi-experimental" designs can be used when randomization is not possible. Each of these designs is associated with different threats to validity.

For instance, perhaps you are in the habit of giving a pre-test and post-test to students to judge how much they learn over the course of the semester. But student learning from other sources—other courses, observations of teaching, from the test itself—cannot be ruled out. One simple solution is including a comparison group in the design. The composition of this group might depend on the research question. For some questions, it might be appropriate to use intending history teachers as a comparison group. For others, it might be better to include intending teachers who have not yet taken the course, students in a calculus course, or mathematics majors.

A longitudinal design allows evidence to be collected at several times. Teachers might, for instance, be tested before a single course, after the course, after a second course, and after they have been teaching for a year. A longitudinal design allows evidence to be collected at several times. Teachers might, for instance, be tested before a single course, after the course, after a second course, and after they have been teaching for a year. The addition of more data, for example, SAT or ACT scores from administrative files, allows for more sophisticated models and inferences about the teachers' growth through the program.

Quasi-experimental designs can also be used to study the effects of a textbook, course, or program. However, logistics are likely to be more complicated than those involved in examining the effect of a course with a given instructor. For example, to separate the effect of a course from the effect of an instructor, multiple instances of the course with different instructors would be a part of the design.

Instrument or measure. Documenting the effect of a treatment requires some sort of measure of that effect. The choice of that measure is likely to depend on which aspects are of interest. Is the treatment a course designed to make teachers become better problem solvers? If so, teachers' problem solving abilities might be measured by a paper and pencil test. Is the treatment a workshop designed to change teachers' beliefs about mathematics? In this case, the teachers might be given a survey, for instance, the ME.ET Beliefs Survey (p. 22).

There are several criteria that are useful to consider when choosing a measure. For example, is the measure aligned to the content of your course or program? If the treatment is an elementary geometry course and you want to know if the teachers have learned any geometry, you may not be interested in a measure with only a few geometry problems.

A test or survey is often the least time-consuming measure to use and interpret. Other methods include structured interviews or analyses of videotaped teaching, e.g., coding for instances of explanations, errors, and other phenomena of interest. However, collecting interview or videotape data often requires more time than administering a test or survey. Interpreting such data can be extremely time consuming—even if they are coded according to a previously designed and tested rubric such as that used for the TIMSS Video Study. If such a rubric is to be used, then time needs to be allowed to train the coders so that the reliability of the results is not compromised.

Key Considerations for Measures

Validity, broadly defined as the extent to which a measure is meaningful, relevant, and useful for the research at hand.

Reliability, broadly defined as the extent to which the measure is free of random error.

Fairness, broadly defined as the extent to which the implementation of the measure is free of systematic error that would undermine validity for one or more subgroups.

Source: *Using Statistics Effectively in Mathematics Education Research*, American Statistical Association: http://www.amstat.org/research_grants/pdfs/SMERReport.pdf

One important decision is whether to use a measure that is designed locally, or to adopt one "off the shelf" from another research project. There are several advantages of using an off-the-shelf measure. One is comparability. Using the same test, survey, or other measure allows the results of a course or program to be easily shared with others involved in the enterprise of educating teachers.

Another advantage of using standardized measures is that they may have been studied with regard to aspects of validity, reliability, and fairness. For example, a study found a positive correlation between students' performance on a commonly used standardized test (the Terra Nova) and their teachers' performance on Learning Mathematics for Teaching (LMT) items.[28] This finding is evidence for LMT's *predictive validity*—whether it predicts performance, in this case, student performance on the Terra Nova.

[28] H. Hill, B. Rowan, & D. Ball, "Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement," *American Educational Research Journal*, 2005. See also, *Measurement*, volume 5, 2007.

Validity is specific to context. It depends on the purpose for which the measure is to be used. For example, the SAT, formerly known as the Scholastic Aptitude Test, was developed for a very specific use—to be used in conjunction with high school grades to predict undergraduate grades. The Educational Testing Service, its developer, has conducted extensive studies of students' SAT scores and their grades. Although the SAT has been used for other purposes, these uses are not backed by corresponding validity studies.

In test construction, there tend to be tradeoffs between the cost of developing the test and its quality. "Quality" includes a multitude of aspects such as validity studies, the test's theoretical underpinnings, and the wording of items. Some of these aspects of assessment were discussed in the 2004 Critical Issues in Mathematics Education Workshop at MSRI and appear in the conference volume *Assessing Mathematical Proficiency*.[29]

Tests of Teacher Knowledge

Learning Mathematics for Teaching (multiple choice):
- Number and operations (K-6, 6–8)
- Patterns, functions, and algebra (K–6, 6–8)
- Geometry (3–8)

www.sitemaker.umich.edu/lmt

Knowledge for Algebra Teaching (multiple choice):
www.educ.msu.edu/kat

Diagnostic Tests for Elementary Teachers (K–5 and 6–8):
- Whole number computation
- Rational number computation
- Geometry and measurement
- Probability, statistics, algebra

http://louisville.edu/edu/crmstd/diag_math_assess_elem_teachers.html

Each test has 20 items: 10 multiple choice and 10 open response.

Interview Protocol

A Study Package for Examining and Tracking Changes in Teachers' Knowledge http://ncrtl.msu.edu/http/tseries/ts931.htm

Rubrics for Analyzing Videotaped Teaching

Mathematical Quality of Instruction Video Codes
http://sitemaker.umich.edu/lmt/faq_about_video_codes

TIMSS 1999 Video Codes
www.llri.org/html/TIMSS/codedev.htm

Beliefs Survey

ME.ET Non-mathematics Items
http://meet.educ.msu.edu/meetinstruments.htm

[29] *Assessing Mathematical Proficiency* can be downloaded at http://www.msri.org/~levy/files/Book53/book53-all.pdf.

Based on a talk by Hyman Bass

Conclusions, Recommendations

Hyman Bass is the Samuel Eilenberg Distinguished University Professor of Mathematics and Mathematics Education at the University of Michigan. He has served as president of the American Mathematical Society, chair of the Mathematical Sciences Education Board at the National Research Council, President of the International Commission on Mathematical Instruction, and on the National Research Council committee that wrote *Adding It Up.*

What have we learned and where do we want to go from here? Important parts of a research and development agenda are:

- identifying the mathematical content, skills, practices, dispositions, and habits of mind that teachers need.
- learning how to teach these mathematical capacities effectively to teachers.
- identifying ways to support instructors and professional developers in this effort.

What to teach. The descriptions of projects and programs for teachers given at the MSRI workshop displayed some common themes. The courses and programs had similar goals. They included mathematical abilities such as procedural fluency, conceptual understanding, problem solving, and reasoning. They included "mathematical habits of mind"—precision, use of representations, and many other mathematical practices. They included "productive disposition"—confidence, interest, and appreciation of mathematics. All of these were mentioned, many of them together, in the workshop presentations.

Another area of agreement was that courses should carefully select content to maximize leverage on the topics that matter for student learning, developing mathematical practices such as representation, definitions, and reasoning as well as fostering productive disposition.

Although courses and programs had common goals, their emphases varied. Some focused on the importance of developing deeper and broader understanding of mathematical topics. Some projects emphasized knowledge of mathematics that is not part of the curriculum that teachers would teach, but was valued as important for building teachers' capacity for teaching. For example, the Vermont Mathematics Initiative and the program at the University of Illinois teach calculus to prospective elementary teachers.

Other projects focused on developing a strong knowledge of the mathematics in children's curriculum from a more advanced standpoint. Still others emphasized the mathematical "habits of mind": precision, careful use of language, and heuristics in problem solving. For instance, Focus on Mathematics, a program for high school and middle school teachers, begins with a "mathematical immersion." In contrast, prospective teachers in UTeach, which prepares middle and high school teachers, begin observations of school classrooms as soon as possible.

How to teach. In discussions of teaching teachers some common themes were emphasis on conceptual understanding and connections among mathematical topics; reasoning and communication, use of multiple representations; and use of student-centered and activity-based learning.

The projects had different entry points and different ways to spark teachers' interest and motivation. Some used case studies, examples of student work, or teaching scenarios as ways to approach teacher learning or the examination of teachers' work. Some contextualized the mathematics assignments, assessments, and choice of representations so that its relevance to practice could be directly seen. Other projects used problems in pure mathematics or formulated problems as they might be seen in academic mathematics courses.

Measuring effectiveness. There were some very articulate, eloquent declarations of what teachers should know. Some speakers argued that teachers should know what is in the school curriculum. Some arguments came solely from the discipline of mathematics, emphasizing the idea that teachers should learn mathematics with mathematical integrity. However, we must be aware that the bottom line is whether knowing mathematics in those ways will produce instruction of higher quality and ultimately improve student learning.

How can we know what we're doing is really effective? Warrants must come from the interpretation of empirical evidence in a manner that allows the field to examine its own practices. Arguments based solely on belief systems and philosophy are not irrelevant, but ultimately we need empirical evidence or the effectiveness of the ways in which we teach mathematics to teachers.

Supporting instructors. There is a variety of ways in which teacher educators can be supported in their work. The presence of incentives and encouragement in mathematics departments and from funders is important. Sometimes there is explicit guidance for instructors in texts or other teaching materials. Sometimes a department provides support for instructors via mentoring and professional development.

Essential intellectual support comes from collaboration with others, particularly those from other disciplines. The teaching of mathematics courses for teachers is a multi-disciplinary undertaking. Educators don't have all the answers, nor do mathematicians, although they play an essential role in the mathematical education of teachers. And cognitive science tells us much about how people learn.

In collaborations between mathematicians and mathematics educators, it may be the case that neither collaborator is fully aware of missing pieces of knowledge that are relevant to educating teachers. At the MSRI workshop, this was illustrated most strongly in descriptions of partnerships and collaborations. Similarly, talks from linguistic, psychometric, and other perspectives illustrated how the social sciences are relevant to the mathematical education of teachers. There is a variety of ways in which language considerations are relevant to what a teacher needs to know about mathematics teaching. Cognitive science shows the importance of measuring the outcome of a course or program. Psychometrics and experimental design provide methods for measuring the outcomes of instruction.

Future work. If we believe that the school system and the learning of children in our schools are important, we need to take a serious professional responsibility for it. This is an enormous task. There are almost four million teachers in the United States. It is unlikely that we can solve the problem of improving teacher knowledge by identifying only exceptional teachers. Instead, the profession of teaching needs to evolve into one in which technical skill can be acquired by ordinary educated adults. Within the United States, this is the case for professions like nursing. In some countries, teaching is such a profession.

To do this, we need a serious professional community of people engaged in the mathematical education of teachers. We need to build a professional infrastructure, treating problems of teacher education in a coherent way and establishing scientific norms and methods. We need to devise ways to accumulate knowledge and create a professional enterprise with standards of scientific rigor.

The MSRI workshop as well as other events suggest that there are seeds of such a professional community which is not yet institutionalized. The workshop participants represent the small pockets of professional work inside departments or inside schools which do not have the critical mass and do not provide the basis for the systemic professional enterprise that is needed. Can this potential coalesce into a powerful and coherent professional community?

Appendix 1: A Guide to the Workshop Sessions

At the MSRI workshop, there were a variety of sessions that described projects concerned with teacher education. These ranged from textbooks, courses, and programs to professional development and beyond. Most addressed specific grade levels and are listed under those headings. The remaining projects appear at the end of this listing.

Each presentation is available as streaming video on the MSRI Web site. To access a particular video, use the link:

www.msri.org/communications/vmath/VMathVideos/VideoInfo/#num/show_video

where #num is the number to the left of the session title.

Elementary and Middle School Teacher Education

Textbooks and Other Curriculum Materials

3222 *Mathematics for Elementary Teachers* by Sybilla Beckmann, published by Addison Wesley.

3228 *Arithmetic for Teachers: With Applications and Topics from Geometry* by Gary Jensen, published by the American Mathematical Society.

3227 *Elementary Mathematics for Teachers* by Thomas Parker and Scott Baldridge, published by Sefton-Ash.

3221 *Reconceptualizing Mathematics* by Judy Sowder, Larry Sowder, and Susan Nickerson, published by W. H. Freeman. Curriculum materials for elementary and middle school teachers.

3263 *Using TIMSS videos for professional development*. The TIMSS videos from 1999 are a rich source of information about teaching in seven countries, and can be used to help teachers learn what students in other countries are learning.
A resource guide (www.rbs.org/international/timss/resource_guide/index.php)has been prepared to help professional developers use these CDs in their work with teachers.

3265 *Developing mathematical ideas*. Discussion of *Measuring Space in One, Two, and Three Dimensions* (http://www2.edc.org/cdt/dmi/dmicur.html) by Deborah Schifter, Susan Jo Russell, and Virginia Bastable, published by Dale Seymour Publications, designed to help K–8 teachers learn the mathematics needed for their teaching.

3264 *Learning and Teaching Linear Functions: Video Cases for Professional Development* by Nanette Seago, Judith Mumme, and Nicholas Branca, published by Heinemann, was created for use by professional development facilitators in their work with grades 6–10 mathematics teachers.

3250 *GeoGebra: Open source software for learning and teaching mathematics* is a free multi-platform software that combines dynamic geometry, algebra and calculus (see http://www.geogebra.org). Its development was part of the Florida Atlantic University and Broward County School Board Mathematics and Science Partnership Institute (session 3235).

Courses and Degree Programs

3213 *The mathematical education of elementary teachers at Delaware*, University of Delaware (www.education.umd.edu/mac-mtl/whatwedo_elementary.htm).

3238 *A mathematics concentration for pre-service elementary school teachers*, K–9 math specialist preparation, University of Illinois, Chicago.

3206 *Collaborative efforts to prepare K–8 mathematics specialists*, statewide preparation program for mathematics specialists, Virginia Mathematics and Science Coalition (www.math.virginia.edu/teach/specialist.htm), see p. 39-41.

3271 *Incorporating English learner strategies in mathematics courses for teachers*, California State Polytechnic University, Pomona.

3209 *Learning mathematical knowledge for elementary school teaching at the University of Michigan*, post-bachelor's degree program for elementary certification (http://mod4.soe.umich.edu/mod4/home).

3204 *The Vermont Mathematics Initiative*, master's program in curriculum and instruction at the University of Vermont for K–8 teachers (www.uvm.edu/~vmi, see p. 36-38).

3219 *A mathematics educator–mathematician partnership for educating teachers at Nebraska*, Math in the Middle, University of Nebraska–Lincoln (http://scimath.unl.edu/MIM/mim.html), see p. 32-34.

3235 *MSP content and technology training for middle grades teachers*. The Mathematics and Science Partnership Institute between Florida Atlantic University and the School Board of Broward County (http://nsfmsp.fau.edu/talks/MSRI/) is designed around a new master's degree program for middle grades teachers offered by the university's department of mathematical sciences.

Professional Development

3240 *Mathematicians Writing for Educators* involves mathematicians writing essays about mathematics for elementary teachers.

3284 *Model drawing: Connecting arithmetic to algebra*, model drawing technique of solving word problems used in the Singapore mathematics curriculum (www.worcester.edu/SMIP/default.aspx), addressed to K–8 teachers.

High School Teacher Education

Courses and Degree Programs

3258 *Content and process in a year-long capstone sequence for secondary teachers*, two-semester course on school mathematics from an advanced viewpoint at Humboldt State University.

3259 *Connecting math major content to high school curriculum*, Rutgers University course for students in the last year of its five-year BA–MEd-certification program for secondary teachers.

3260 *Two non-traditional content courses for in-service high school teachers at the Harvard Extension School*, one course in geometry taught in the spirit of the Core Plus and Connected Mathematics Project curricula, probability and combinatorics course taught via the Moore Method.

3273 *Using extended mathematics tasks so as to increase teachers' mathematical knowledge for teaching*, Ann Shannon.

3256 *Teaching teachers in a research mathematics department*, University of California at Los Angeles program for teaching future mathematics teachers. The main components of the program include a yearlong hybrid capstone/methods course for seniors interested in teaching and a "mathematics for teaching" major.

3230 *UTeach at the University of Texas*, collaborative program of the Colleges of Natural Sciences and Education at the University of Texas at Austin and the Austin Independent School District (www.uteach.utexas.edu).

3233 *MfA: A replicable NYC program*. The five-year Math for America Fellowship program (www.mathforamerica.org/home) is designed to attract, train and retain outstanding public secondary school math teachers. MfA also has programs in Los Angeles, San Diego, and Washington, DC.

3247 *The impact of immersion in mathematics on teachers*, an "immersion" approach to professional development, developed and refined in programs like PROMYS for Teachers (www.promys.org), Focus on Mathematics (www.focusonmath.org), and the Park City Mathematics Institute (http://mathforum.org/pcmi).

Other Projects

3274 *What French didactique can say to American mathematics educators*, Virginia Warfield (University of Washington discusses her work with the French education researcher Guy Brousseau.

3255 *Learning together: Implementing inquiry-based content courses for K–20 teachers of mathematics*, Ruth Parker (Mathematics Education Collaborative) discusses the work of the Greater Birmingham Mathematics Partnership.

3223 *Mentoring faculty and graduate students interested in teaching teachers*. The University of Georgia is developing a program to prepare graduate students and postdoctoral fellows to teach courses for prospective elementary teachers.

Appendix 2

Reports on Teacher Education in Chronological Order

Leitzel, J. R. C. (Ed.). (1991). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics.* Washington, DC: Mathematical Association of America.

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Cochran-Smith, M. & Zeichner, K. M. (Eds.). (2005). *Studying teacher education: The report of the American Educational Research Association panel on research and teacher education.* Hillsdale, NJ: Lawrence Erlbaum Associates.

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