MSRI2018, LECTURE 1

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1. $\bar{\partial}$ method and Hartogs extension

We begin in complex euclidean space $\mathbb{C}^n$, with standard coordinates $(z_1, \dots z_n)$, $z_k \in \mathbb{C}$ and $z_k = x_k + iy_k$ for $k = 1, \dots n$.

1.1. **Cauchy-Riemann complex I.** Let $\Omega \subset \mathbb{C}^n$ be a *domain*, i.e. an open, connected set. If $f : \Omega \longrightarrow \mathbb{C}$ and $f \in C^1(\Omega)$ (see Function Space table), define

$$\begin{aligned} \frac{\partial f}{\partial z_k} =& \frac{1}{2}\left[\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\right] \\ \frac{\partial f}{\partial \bar{z}_k} =& \frac{1}{2}\left[\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right], \qquad k = 1, \dots, n. \end{aligned} \tag{1.1}$$

The equations (1.1) define the first-order differential operators $\frac{\partial}{\partial z_k}$ and $\frac{\partial}{\partial \bar{z}_k}$. These operators are called the Cauchy-Riemann vector-fields and are easily seen to be *derivations*; thus we also refer to these operators as *tangent vectors* on $\mathbb{C}^n$. If we consider the coordinate functions $e_k(z_1, \dots z_n) = z_k$ and their conjugates $\bar{e}_k(z_1, \dots z_n) = \bar{z}_k$, notice

$$\frac{\partial}{\partial z_k}(e_\ell) = \delta_{kl} = \frac{\partial}{\partial \bar{z}_k}(\bar{e}_\ell), \quad 1 \le k, l \le n$$

where δ_{kl} is the Kronecker symbol, while

$$\frac{\partial}{\partial z_k}(\bar{e}_m) = 0 = \frac{\partial}{\partial \bar{z}_k}(\bar{e}_m), \quad 1 \le k, m \le n.$$

Differential forms, or *co-tangent vectors*, connected to the complex structure are also important. The basic first-order ones are

$$dz_k = dx_k + idy_k \quad \text{and} \quad d\bar{z}_k = dx_k - idy_k, \qquad k = 1, \dots, n.$$

Tangent vectors and co-tangent vectors naturally "pair" with each other. This is elaborated later. For now, take the view that $\{dz_k, d\bar{z}_k\}$ are basis elements of an abstract vector space. When a function is written adjacent to one of these symbols, the understanding is the function is evaluated at a point and this number multiplies the corresponding symbol. Since the evaluation is often not explicitly written, this may be confusing initially but becomes familiar with practice.

For $f \in C^1(\Omega)$, the *exterior derivative* of f is defined

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} dy_j.$$

Some algebra allows this to be re-written as

$$\begin{aligned} df &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \\ &= \partial f + \bar{\partial} f. \qquad \textit{ExerciseI.} \end{aligned}$$

The last equation defines the operators ∂ and $\bar{\partial}$, i.e.,

$$\bar{\partial} f(z) = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(z) d\bar{z}_j \quad \text{for } z \in \Omega \tag{1.2}$$

and similarly for ∂f. Thus $\bar{\partial} f$ is a linear combination of the differential forms $d\bar{z}_j$. In this way, it is an example of a *(0,1)-form*.

1.2. **Holomorphic functions.**

Definition 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $f \in C^1(\Omega)$ is* **holomorphic** *on Ω if $\bar{\partial} f(z) = 0$ for all $z \in \Omega$. Let $\mathcal{O}(\Omega)$ denote the set of holomorphic functions on Ω.*

It's important to remember that $\bar{\partial} f(z) = 0$, which looks like a scalar equation, is actually a system of equations: $\bar{\partial} f = 0$ means $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all $j = 1, \dots n$.

The equation $\frac{\partial f}{\partial \bar{z}_j}(z) = 0$, for some j, says that f satisfies the one-variable Cauchy-Riemann equations in the complex variable z_j ExerciseI. Since we are assuming $f \in C^1(\Omega)$, it follows that f is holomorphic in z_j. Thus Definition 1.3 says every $f \in \mathcal{O}(\Omega)$ is holomorphic as a function of one complex variable separately in each of the underlying variables $z_1, \dots z_n$.

A theorem of Hartogs says that a function $f(z_1, \dots, z_n)$ that is holomorphic separately in each z_j must also satisfy Definition 1.3, i.e. the condition $f \in C^1(\Omega)$ necessarily follows. This is false for smooth functions as the example $g(x, y) = \frac{xy}{x^2+y^2}$ in $\mathbb{R}^2$ shows. The proof of Hartogs theorem is outside the main theme of these lectures, but will be discussed in the afternoon sessions.

1.3. **Cauchy-Riemann complex II.**

A (p, q)-form, for p and q positive integers such that $1 \le p, q \le n$, is obtained by taking wedge products of the differentials dz_k and $d\bar{z}_k$, with exactly p factors of the various dz_js and q factors of the $d\bar{z}_j$s. Multi-index notation is useful: if $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots j_q)$ are p and q-tuples respectively, write

$$dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \quad \& \quad d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

and denote $p = |I|$, $q = |J|$.

A (p, q)-form is a linear combination of $dz^I \wedge d\bar{z}^J$, as I and J range over all indices such that $|I| = p$ and $|J| = q$, i.e.

$$\beta = \sum_{\substack{|I|=p \\ |J|=q}} \beta_{IJ}\, dz^I \wedge d\bar{z}^J \tag{1.4}$$

is a (p, q)-form, where the coefficients β_{IJ} are functions on Ω.

Classes of (p, q)-forms are obtained by requiring each β_{IJ} to belong to certain function spaces. Initially the most relevant classes of (p, q)-forms are those with coefficients in $C^\infty(\Omega)$, $C^\infty\left(\overline{\Omega}\right)$, or $C_0^\infty(\Omega)$, which we'll denote as

$$\Lambda^{p,q}(\Omega), \quad \Lambda^{p,q}\left(\overline{\Omega}\right), \quad \text{or} \quad \Lambda_0^{p,q}(\Omega)$$

respectively.

1.4. **The $\bar\partial$ method.** The main analytic problem in several complex variables is the *construction* of functions $h \in \mathcal{O}(\Omega)$ with various side properties $\mathcal{P}$. There are many interesting $\mathcal{P}$, some examples are: taking given values on prescribed sub-varieties of Ω, blowing up as $z \to p \in b\Omega$, and being metrically or norm dominated by a given expression.

Power series methods, which are very successful in one-variable, generally fail in $\mathbb{C}^n, n \geq 2$. A powerful several variable technique for construction in several variables is to consider the inhomogeneous $\bar\partial$ system.

The method:

(a) Construct $f \in C^\infty(\Omega)$ with the desired property $\mathcal{P}$ — this is usually easy.
(b) Consider the $(0,1)$-form $\alpha =: \bar\partial f$.
(c) Find a solution to $\bar\partial u = \alpha$ *that is well-behaved wrt property* $\mathcal{P}$.
(d) Define $h =: f - u$. Then $h \in \mathcal{O}(\Omega)$ and satisfies $\mathcal{P}$.

The italicized phrase suggests both the scope and challenge of the method. Its precise formulation depends on the character of property $\mathcal{P}$. We turn to our first example in the next section.

1.4.1. *$\bar\partial$ is over-determined.* The method requires solving $\bar\partial u = \alpha$, for α a given $(0,1)$-form. Or, in more general situations, if α is a (p,q)-form. For the moment, consider forms with smooth coefficients, i.e., in $\Lambda^{p,q}(\Omega)$, to focus on the main issue. If $\alpha \in \Lambda^{0,1}(\Omega)$, the equation $\bar\partial u = \alpha$ is *over-determined* in $\mathbb{C}^n$, $n > 1$.

$$\bar\partial u = \alpha \quad \Leftrightarrow \quad \begin{cases} \frac{\partial u}{\partial \bar z_1}(z) = \alpha_1(z) \\ \qquad \vdots \\ \frac{\partial u}{\partial \bar z_n}(z) = \alpha_n(z) \end{cases}$$

Notice there are n data functions – $\alpha_1, \ldots, \alpha_n$ – but only a single solution – u. Therefore α must satisfy *compatibility conditions* before there is a chance to solve $\bar\partial u = \alpha$. Indeed if u solves $\bar\partial u = \alpha$, then

$$\frac{\partial \alpha_k}{\partial \bar z_j} = \frac{\partial^2 u}{\partial \bar z_j \partial \bar z_k} = \frac{\partial \alpha_j}{\partial \bar z_k}, \qquad \forall k, j. \tag{1.5}$$

Consequently only α satisfying the necessary conditions (1.5) are allowed when trying to solve $\bar\partial u = \alpha$. The conditions (1.5) are more succinctly expressed after the $\bar\partial$ operator is extended to higher level forms in Lecture 3.

1.5. **Solving $\bar\partial$ with compact support.** Our first example of an interesting side condition is to take $\mathcal{P} =$ "has compact support".

Theorem 1.6 (Solving $\bar\partial$ with compact support)**.** *If $\alpha \in \Lambda_0^{0,1}(\mathbb{C}^n)$, $n > 1$, satisfies conditions* (1.5)*, there exists $u \in \Lambda_0^{0,0}(\mathbb{C}^n)$ such that $\bar\partial u = \alpha$.*

Remark 1.7. This result does not hold in $\mathbb{C}^1$, unless $\int_{\mathbb{C}} \alpha(\zeta)\zeta^k \, dV(\zeta) = 0$ for all $k \in \mathbb{Z}^+$. ExerciseII.

The proof of Theorem 1.6 uses the one-variable *generalized* Cauchy Integral formula. This result is important enough to state separately

Theorem 1.8 (Generalized CIF)**.** *If $\Omega \subset \mathbb{C}$ is a domain with C^1 boundary (see Lecture 5) and $u \in C^1\left(\bar{\Omega}\right)$, then*

$$u(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{\Omega} \frac{\partial u/\partial\bar{\zeta}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}, \qquad z \in \Omega. \tag{1.9}$$

Proof. Fix $z \in \Omega$. For $\epsilon > 0$, let $B(z,\epsilon) = \{w \in \mathbb{C} : |w - z| < \epsilon\}$ be the disc centered at z of radius ϵ. Consider the domain $\Omega_\epsilon =: \Omega \setminus \overline{B(z,\epsilon)}$.

Recall Stokes theorem: if D is a C^1 bounded domain and $\omega \in \Lambda^{1,0}\left(\overline{D}\right)$ then

$$\int_{bD} \omega = \int_D d\omega,$$

where bD is positively oriented. Apply Stokes to $\omega(\zeta) = \frac{u(\zeta)}{\zeta - z}\, d\zeta$ on the domain Ω_ϵ, noting the two separate components, $b\Omega$ and $bB(z,\epsilon)$, of the boundary $b\Omega_\epsilon$. The result is

$$\int_{b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta - i\int_0^{2\pi} u\left(z + \epsilon e^{i\theta}\right)\, d\theta = \int_{\Omega_\epsilon} \frac{\partial u/\partial\bar{\zeta}}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta.$$

The second integral on the LHS is $\{b\mathbb{B}(z,\epsilon)\}\omega$, parameterizing by $\zeta(\theta) = z + \epsilon e^{i\theta}$ and the $-$ sign coming from positively orienting $b\Omega_\epsilon$. Rearranging this equation, using $d\bar{\zeta} \wedge d\zeta = -d\zeta \wedge d\bar{\zeta}$,

$$\int_{b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta + \int_{\Omega_\epsilon} \frac{\partial u/\partial\bar{\zeta}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = i\int_0^{2\pi} u\left(z + \epsilon e^{i\theta}\right)\, d\theta.$$

Let $\epsilon \to 0$. The RHS tends to $2\pi i u(z)$, since u is continuous. This gives (1.9). □

Note that when $u \in \mathcal{O}(\Omega)$, Theorem 1.8 reduces to the usual Cauchy Integral formula.

Proof of Theorem 1.6. Define

$$\begin{aligned} u(z_1, z_2, \dots z_n)) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\alpha_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\alpha_1(\zeta + z_1, z_2, \dots, z_n)}{\zeta}\, d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

The second equality follows by change of variables $\zeta \to \zeta + z_1$.

Consider the second integral defining u. Differentiation under the integral sign shows $u \in C^\infty(\mathbb{C})$. Additionally, taking $\frac{\partial}{\partial\bar{z}_k}$, for any $k = 1, \dots, n$, of both sides yields

$$\begin{aligned} \frac{\partial u}{\partial\bar{z}_k} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial\alpha_1}{\partial\bar{z}_k}(\zeta + z_1, z_2, \dots z_n)}{\zeta}\, d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial\alpha_k}{\partial\bar{z}_1}(\zeta + z_1, z_2, \dots z_n)}{\zeta}\, d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial\alpha_k}{\partial\bar{z}_1}(\zeta, z_2, \dots z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta} \end{aligned} \tag{1.10}$$

The compatibility conditions (1.5) give the second equality. The third equality follows by change of variables.

Choose R large enough so that supp $\alpha \subset B(0,R)$. In particular $\alpha \equiv 0$ on $bB(0,R)$. Theorem 1.8 then says that

$$\frac{1}{2\pi i}\int_{B(0,R)} \frac{\frac{\partial\alpha_k}{\partial\bar{z}_1}(\zeta, z_2, \dots z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta} = \alpha_k(z_1, \dots, z_n),$$

But the LHS above equals the RHS of (1.10). Thus $\frac{\partial u}{\partial \bar{z}_k} = \alpha_k$ $k = 1, \ldots, n$; in other words $\bar{\partial} u = \alpha$. $\square$

1.6. **Hartogs extension.** Theorem 1.6 leads to a remarkable extension result proved by Hartogs (by a different method) more than 100 years ago. The phenomena expressed by this result has no analog in $\mathbb{C}^1$. This result (and the biholomorphic inequivalence of topological cells in $\mathbb{C}^n$) inaugurated the study of several complex variables as a separate field.

Theorem 1.11 (Hartogs)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$, $n > 1$, and $K \subset\subset \Omega$ such that $\Omega \setminus K$ is connected.*

Then every $f \in \mathcal{O}(\Omega \setminus K)$ is the restriction of an $F \in \mathcal{O}(\Omega)$.

Let's sketch how the $\bar{\partial}$ method works here:

(a) Let $\chi \in C_0^\infty(\Omega)$ such that $\chi \equiv 1$ on a neighborhood of K. Set

$$\mathbf{f}(z) = \begin{cases} [(1-\chi)\cdot f](z) & \text{if } z \in \Omega \setminus K \\ 0 & \text{if } z \in K \end{cases}$$

(b) Set $\alpha = \bar{\partial}\mathbf{f}$. Note that $\alpha = -\left(\bar{\partial}\chi\right) \cdot f$ where f is defined, $\bar{\partial}\alpha = 0$, and $\alpha \in \Lambda_0^{0,1}(\mathbb{C}^n)$ (just extend α by 0 off $\operatorname{supp}\left(\bar{\partial}\chi\right)$).

(c) Invoke Theorem 1.6 to find $u \in \Lambda_0^{0,0}(\mathbb{C}^n)$ solving $\bar{\partial} u = \alpha$.

(d) Define $F = \mathbf{f} - u$. Consider the component of $b\Omega$ that also bounds a neighborhood of ∞. In an open set of this boundary component, $u \equiv 0$ and $\chi \equiv 0$. Thus, $F = f$ on this set. Since $\Omega \setminus K$ is connected, $F = f$ on all of $\Omega \setminus K$.

A more formal proof of Theorem 1.11 can be found in Krantz, "Function theory of several complex variables", Theorem 1.2.6.