MSRI2018, LECTURE 6

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Let $\Omega \subset \mathbb{C}^n$ be a domain. Recall from Section 1.4 of Lecture 1, that the $\bar{\partial}$ method consists of four steps, the third of which involves finding solutions to the equation

$$\bar{\partial} u = \alpha, \tag{0.1}$$

where $\alpha \in \bar{\partial} \in \Lambda^{p,q}(\Omega)$ is a closed (p,q)-form with coefficients in $C^\infty(\Omega)$. The proof of existence of solutions to equation (0.1) requires tools from functional analysis which we develop now.

Theorem 0.2 (Riesz representation theorem for Hilbert spaces)**.** *Let λ be a bounded linear functional on a Hilbert space H. Then there exists a unique $g \in H$ such that*

$$\lambda(f) = (f, g) \quad \textit{for all } f \in H.$$

Moreover, $\|g\| = \|\lambda\|$.

Proof. See [**?**, Theorem 5.3]. □

1. Closed operators

An *operator* T from a Hilbert space H_1 to another Hilbert space H_2 is a $\mathbb{C}$-linear map from a linear subspace $\mathrm{Dom}(T)$ of H_1 into H_2. If $\mathrm{Dom}(T)$ is dense in H_1, we say that T is *densely defined*. We will assume that all our operators are densely defined. The subspace $\mathrm{Dom}(T)$ is called the *domain* of T. Recall that an operator $T : H_1 \to H_2$ is said to be bounded if there is a $C > 0$ such that for all $f \in \mathrm{Dom}(T)$ we have

$$\|Tf\|_{H_2} \leq C \|f\|_{H_1}.$$

The analysis of unbounded operators becomes interesting if a densely defined unbounded operator is also *closed*.

Definition 1.1. *A linear operator $T : H_1 \dashrightarrow H_2$ is* closed *if its graph $\Gamma(T)$*

$$\Gamma(T) = \{(f, Tf) \in H_1 \times H_2 | f \in Dom(T)\}$$

is a closed subspace of $H_1 \times H_2$, where $H_1 \times H_2$ is given the product topology, which is induced by the norm:

$$\|(h_1, h_2)\|^2_{H_1 \times H_2} = \|h_1\|^2_{H_1} + \|h_2\|^2_{H_2}.$$

It is clear that this definition can be reformulated as below.

Proposition 1.2. *The following are equivalent:*

1. *A linear operator $T : H_1 \dashrightarrow H_2$ is closed.*
2. *For every sequence (f_n) in $Dom(T)$ such that $f_n \to f$ in H_1 and $Tf_n \to g$ in H_2, we have $f \in Dom(T)$ and $g = Tf$.*

Recall the following fundamental result from functional analysis.

Theorem 1.3 (Closed Graph Theoem)**.** *A linear operator $T : H_1 \to H_2$ defined on all of H_1 is closed iff it is continuous.*

Definition 1.4 (Adjoint of an operator)**.** *Let $T : H_1 \dashrightarrow H_2$ be an operator. The* adjoint *$T^* : H_2 \dashrightarrow H_1$ of T is the operator with*

$$Dom(T^*) = \{g \in H_2 | f \mapsto (Tf, g) \text{ is a bounded functional for all } f \in Dom(T)\}$$

and satisfying

$$(Tf, g)_{H_2} = (f, T^* g)_{H_1}$$

for all $f \in Dom(T)$ and $g \in Dom(T^*)$.

The existence of the adjoint follows from the Riesz representation theorem. Exercise.
Exercise III. Let $T : H_1 \dashrightarrow H_2$ be a densely defined operator. Show that T^* is closed.

Lemma 1.5. *Let $T : H_1 \dashrightarrow H_2$ be an operator. Then*

(1) $(\ker(T))^\perp = \overline{Range(T^*)}$
(2) $\overline{Range(T)} = (\ker(T^*))^\perp$

Proof. Exercise IV. □

We now turn to the class of unbounded operators that we are interested in.

2. Differential operators

Let $\Omega \subset \mathbb{R}^n$ be a domain. We equip the space $C_0^\infty(\Omega)$ with the following inner product

$$(f, g) = \int_\Omega f \bar{g} dV \quad \text{for all } f, g \in C_0^\infty(\Omega). \tag{2.1}$$

A *differential operator* $A : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ of order m with coefficients in $C^\infty(\Omega)$ is given by its action on a function u as

$$Au = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u, \tag{2.2}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We assume that at least one function a_α with $|\alpha| = m$ does not vanish identically.

We want to think of these differential operators as operators acting on $L^2(\Omega)$ or perhaps some weighted L^2-space. Such a "Hilbert space realization" of a differential operator is never bounded, provided the order m of the operator is at least 1. Instead of giving a general proof of this, let us illustrate why this is the case using an example. Let the operator $D : L^2(0,1) \dashrightarrow L^2(0,1)$ with $\mathrm{Dom}(D) = C_0^\infty(0,1)$ be given by

$$Df(x) = f'(x) \quad \text{for all } x \in (0,1).$$

Then if $\phi \in \mathcal{C}_0^\infty((0,1))$ is a nonzero compactly supported smooth function, and we set $\phi_\epsilon(x) = \epsilon^{-\frac{1}{2}} \phi(\frac{x}{\epsilon})$, we have that $\|\phi_\epsilon\|$ is independent of ϵ, but $\|D\phi_\epsilon\| \to \infty$ as $\epsilon \to 0$. This idea easily generalizes to arbitrary differential operators of order ≥ 1.

2.1. Formal adjoint of a differential operator.

Definition 2.3. *Given a differential operator $A : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$, let $A' : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ be the operator satisfying*

$$(Af, g) = (f, A'g) \quad \text{for all } f, g \in C_0^\infty(\Omega). \tag{2.4}$$

The operator A' is said to be the formal adjoint *of the differential operator A.*

We now show that the formal adjoint A' of a differential operator A of order m exists and is a differential operator of order m. Let A be as in (2.2). For functions $f, g \in C_0^\infty(\Omega)$ we obtain the following identity by integrating by parts

$$\int_\Omega (a_\alpha D^\alpha f)\bar{g} dV = (-1)^{|\alpha|} \int_\Omega f \overline{D^\alpha(\bar{a}_\alpha g)} dV, \tag{2.5}$$

where α is a multi-index with $|\alpha| \leq m$. Let $A' : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ be the differential operator given by

$$A'u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha(\bar{a}_\alpha(x))u \quad \text{for all } u \in C_0^\infty(\Omega),$$

where the multi-indices α and the coefficients a_α are as in (2.2). In view of equation (2.5) we see that the operator A' is the formal adjoint of the differential operator A in (2.2).

Exercise V. Let $\Omega \subset \mathbb{R}^n$ be a domain with C^1 boundary. Let $(\nu_1, \ldots, \nu_n)$ be the outward pointing unit vector to $b\Omega$ and $d\sigma$ be the surface measure. Show that

$$\int_\Omega u \frac{\partial v}{\partial x_j} dV = \int_{b\Omega} uv\nu_j d\sigma - \int_\Omega v \frac{\partial u}{\partial x_j} dV.$$

2.2. **Realizations of differential operators.** Let Ω be a domain in $\mathbb{R}^n$ and $A : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ be a differential operator. The operator A gives rise to several closed operators on the Hilbert space $L^2(\Omega)$ two of which are given below.

Definition 2.6 (Weak maximal realization)**.** *For $f, g \in L^2(\Omega)$ we say that $Af = g$* weakly *if for all $\phi \in C_0^\infty(\Omega)$ we have*

$$(f, A'\phi) = (g, \phi).$$

The weak maximal realization $A_{\max} : L^2(\Omega) \dashrightarrow L^2(\Omega)$ *associated to A has domain*

$$Dom(A_{\max}) = \{f \in L^2(\Omega) | \exists\, g \in L^2(\Omega) \text{ such that } Af = g \text{ weakly}\}$$

and $A_{\max} f = Af$ weakly.

Definition 2.7 (Strong minimal realization)**.** *The* strong minimal realization $A_{\min} : L^2(\Omega) \dashrightarrow L^2(\Omega)$ *is the closure of the densely defined operator $A|_{C_0^\infty(\Omega)}$ on $L^2(\Omega)$ in the graph topology. That is, $Dom(A_{\min})$ consists of those $f \in L^2(\Omega)$ for which there is a $g \in L^2(\Omega)$ and a sequence $(f_n) \subset C_0^\infty(\Omega)$ such that $f_n \to f$ and $Af_n \to g$ in $L^2(\Omega)$.*

More generally differential operators may act on sections of vector bundles. The $\bar{\partial}$ operator for example, acts on exterior powers of the cotangent bundle of a complex manifold. Then the operator $\bar{\partial}$ is locally given by a matrix of partial differential operators. Consider the following example.

Let Ω be a domain in $\mathbb{C}^n$. Then $\Lambda^{0,1}(\Omega) \cong (C^\infty(\Omega))^n$ and let $d\bar{z}_1, \ldots, d\bar{z}_n$ be the ordered basis for $\Lambda^{0,1}(\Omega)$. Then the operator $\bar{\partial} : C^\infty(\Omega) \to \Lambda^{0,1}(\Omega)$ is given by the matrix

$$\bar{\partial} f = \left[\frac{\partial}{\partial \bar{z}_1} \cdots \frac{\partial}{\partial \bar{z}_n} \right] f \quad \text{for all } f \in C^\infty(\Omega).$$

Lemma 2.8. *The operators $A_{\max} : L^2(\Omega) \dashrightarrow L^2(\Omega)$ and $A'_{\min} : L^2(\Omega) \dashrightarrow L^2(\Omega)$ are adjoints of each other.*

Proof. See [**?**, Lemma 4.3]. □

3. Domains of Hilbert space adjoints

Let us do a computation to understand the domain of an adjoint. Consider the operator

$$D = \left(\frac{d}{dx}\right)_{\max} : L^2(0,1) \dashrightarrow L^2(0,1)$$

which is the maximal realization of the differential operator $\frac{d}{dx}$ acting on functions on the interval $(0,1)$. Assume that $u, v \in \mathcal{C}^1([0,1])$, and note that

$$(Du, v)_{L^2} = \int_0^1 u'(x)v(x)dx = \int_0^1 u(x)\left(-v'(x)\right)dx + \left(u(1)v(1) - u(0)v(0)\right).$$

It is easy to see that the map $u \mapsto u(1)v(1) - u(0)v(0)$ is bounded in the norm of $L^2(0,1)$ if and only if $v(1) = v(0) = 0$. On the other hand the map

$$u \mapsto \int_0^1 u(x)\left(-v'(x)\right)dx$$

is bounded. Therefore we conclude that

$$\mathrm{Dom}(D^*) \cap \mathcal{C}^1([0,1]) = \{v \in \mathcal{C}^1([0,1]) | v(1) = v(0) = 0\},$$

i.e., the domain of the adjoint is subject to certain *boundary conditions*. As we will see later, this fact plays a crucial role.

4. L^2 existence theorem for $\bar{\partial}$

Lemma 4.1. *Let $T : H_1 \to H_2$ be a linear, closed, densely defined operator. The following conditions on T are equivalent.*

(1) Range(T) is closed.
(2) There is a constant $C > 0$ such that

$$\|f\|_{H_1} \leq C\,\|Tf\|_{H_2} \quad \textit{for all } f \in \textit{Dom}(T) \cap \overline{\textit{Range}(T^*)}.$$

(3) Range(T^*) is closed.
(4) There is a constant $C > 0$ such that

$$\|f\|_{H_2} \leq C\,\|T^*f\|_{H_1} \quad \textit{for all } f \in \textit{Dom}(T^*) \cap \overline{\textit{Range}(T)}.$$

The best constants in (2) and (4) are the same.

Proof. See [**?**, Lemma 4.1.1.]. □

4.1. **$\bar{\partial}$ as a Hilbert space operator.** Let Ω be a bounded domain in $\mathbb{C}^n$ and $\varphi : \Omega \to \mathbb{R}$ be a strictly plurisubharmonic function. Let $L^2_{0,1}(\Omega, e^{-\varphi})$ be the Hilbert space

$$L^2_{0,1}(\Omega, e^{-\varphi}) = \left\{ f = \sum_{j=1}^n f_j d\overline{z}_j \;\middle|\; \sum_{j=1}^n \int_\Omega |f_j|^2 e^{-\varphi} dV < \infty \right\}.$$

The space $L^2_{0,1}(\Omega, e^{-\varphi})$ is equipped with the following inner product

$$(f, g)_\varphi = \sum_{j=1}^n \int_\Omega f_j \overline{g_j} e^{-\varphi} dV \quad \text{for all } f, g \in L^2_{0,1}(\Omega, e^{-\varphi}).$$

By abuse of notation, we denote the weak maximal realization of the differential operator $\bar{\partial} : \Lambda(\Omega) \to \Lambda^{0,1}(\Omega)$ by $\bar{\partial} : L^2(\Omega, e^{-\varphi}) \dashrightarrow L^2_{0,1}(\Omega, e^{-\varphi})$, the latter being a Hilbert space operator. We denote by $\bar{\partial}^\star_\varphi : L^2_{0,1}(\Omega, e^{-\varphi}) \dashrightarrow L^2(\Omega, e^{-\varphi})$ the Hilbert space adjoint of the operator $\bar{\partial}$.

We now present the main application of functional analysis in the proof of Hörmander's theorem.

Theorem 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $\phi \in \mathcal{C}^1(\overline{\Omega})$. That there is a constant $K > 0$ such that for every $(0,1)$-form $g \in Dom(\bar\partial) \cap Dom(\bar\partial^\star_\varphi)$ the following inequality holds*

$$\left\|\bar\partial g\right\|^2_\varphi + \left\|\bar\partial^\star_\varphi g\right\|^2_\varphi \geq K^2 \left\|g\right\|^2_\varphi \tag{4.3}$$

Then for all $f \in L^2_{0,1}(\Omega)$ with $\bar\partial f = 0$, there is a $u \in L^2(\Omega)$ such that

$$\bar\partial u = f$$

and

$$\left\|u\right\|_\phi \leq \frac{1}{K}\left\|g\right\|_\phi$$

*Remark.*It will be shown that inequality (4.3) holds for a smoothly bounded pseudoconvex domain Ω is ϕ is a strictly psh function which is $\mathcal{C}^2$ in Ω, and K is such that $i\partial\bar\partial\phi \geq K$. This is the *basic estimate* of L^2 theory who proof will take up most of the remaining week.

Proof. Since $\bar\partial^2 = 0$, we have $(\bar\partial^\star_\varphi)^2 = 0$ and hence

$$\overline{\mathrm{Range}(\bar\partial)} \subset \ker(\bar\partial) \text{ and} \tag{4.4}$$

$$\overline{\mathrm{Range}(\bar\partial^\star_\varphi)} \subset \ker(\bar\partial^\star_\varphi), \tag{4.5}$$

since $\ker(\bar\partial)$ and $\ker(\bar\partial^\star_\varphi)$ are closed subspaces of $L^2(\Omega, e^{-\varphi})$ and $L^2_{0,1}(\Omega, e^{-\varphi})$ respectively.

It follows from (4.3) that

$$\left\|\bar\partial^\star_\varphi g\right\|^2_\varphi \geq K^2 \left\|g\right\|^2_\varphi \tag{4.6}$$

for all $(0,1)$-forms $g \in \mathrm{Dom}(\bar\partial^\star_\varphi) \cap \ker(\bar\partial)$.

By equation (4.5), the inequality (4.6) holds for all $g \in \mathrm{Dom}(\bar\partial^\star_\varphi) \cap \overline{\mathrm{Range}(\bar\partial)}$ and applying Lemma 4.1 to the densely defined, closed operator $\bar\partial : L^2(\Omega, e^{-\varphi}) \dashrightarrow L^2_{0,1}(\Omega, e^{-\varphi})$ we get that $\mathrm{Range}(\bar\partial^\star_\varphi)$ is closed in $L^2(\Omega, e^{-\varphi})$ and $\mathrm{Range}(\bar\partial)$ is closed in $L^2_{0,1}(\Omega, e^{-\varphi})$.

Let $L : \mathrm{Range}(\bar\partial^\star_\varphi) \to \mathbb{C}$ be the linear functional given by

$$L(\bar\partial^\star_\varphi g) = (g, f)_\varphi \text{ for all } g \in \mathrm{Dom}(\bar\partial^\star_\varphi).$$

We now show that L is a bounded operator. Let $g \in \mathrm{Dom}(\bar\partial^\star_\varphi)$ be given. Then g can be written as $g = g_1 + g_2$ where $g_1 \in \mathrm{Dom}(\bar\partial^\star_\varphi) \cap \ker(\bar\partial)$ and $g_2 \in \mathrm{Dom}(\bar\partial^\star_\varphi) \cap (\ker(\bar\partial))^\perp$. Then we have

$$|(f, g_1)_\varphi| \leq \left\|f\right\|_\varphi \left\|g_1\right\|_\varphi \leq \frac{1}{K}\left\|f\right\|_\varphi \left\|\bar\partial^\star_\varphi g_1\right\|_\varphi.$$

Also, $(f, g_2)_\varphi = 0$, since $f \in \ker(\bar\partial)$ and $g_2 \in (\ker\bar\partial)^\perp$. Also $\bar\partial^\star_\varphi g_2 = 0$ since $(\ker(\bar\partial))^\perp = \overline{\mathrm{Range}(\bar\partial^\star_\varphi)} = \mathrm{Range}(\bar\partial^\star_\varphi)$. Thus, we have

$$\begin{aligned} |(f, g)_\varphi| &= |(f, g_1)|_\varphi \\ &\leq \frac{1}{K}\left\|f\right\|_\varphi \left\|\bar\partial^\star_\varphi g_1\right\|_\varphi \\ &= \frac{1}{K}\left\|f\right\|_\varphi \left\|\bar\partial^\star_\varphi g\right\|_\varphi. \end{aligned}$$

This shows that $\|L\| \leq \frac{1}{K} \cdot \left\|f\right\|_\varphi$ and hence L is bounded. By Hahn-Banach theorem, L extends to a bounded linear functional $\widetilde{L}$ defined on all of $L^2(\Omega, e^{-\varphi})$ such that $\left\|\widetilde{L}\right\| = \|L\|$.

Then, by the Riesz Representation theorem, there is a $u \in L^2(\Omega, e^{-\varphi})$ such that for all $g \in \mathrm{Dom}(\bar{\partial}^\star_\varphi)$ we have

$$(\bar{\partial}^\star_\varphi g, f)_\phi = L(\bar{\partial}^\star_\varphi g) = \widetilde{L}(\bar{\partial}^\star_\varphi g) = (\bar{\partial}^\star_\varphi g, u)_\varphi \tag{4.7}$$

and

$$\|u\|_\varphi = \left\|\widetilde{L}\right\| = \|L\| \leq \frac{1}{K} \|f\|_\varphi . \tag{4.8}$$

It follows from equation (4.7) that $\bar{\partial} u = f$ weakly. $\square$