Exercises for Lecture 1

(1) Let $\mathbb{C}^{n-2} \subset \mathbb{C}^n$ be the linear subspace spanned by the first $n-2$ coordinates. Show that any holomorphic function $f : \mathbb{C}^n \setminus \mathbb{C}^{n-2} \to \mathbb{C}$ admits a holomorphic extension $\hat{f} : \mathbb{C}^n \to \mathbb{C}$.

(2) Let f be a holomorphic function on $\mathbb{C}^n$. What can you say about the $(1,1)$-form $\partial\bar{\partial}\log|f|^2$?

(3) Study the analytic sets $S_1 = \{z^2(z+1) - w^2 = 0\}$ and $S_2 = \{z^3 - w^2 = 0\}$ in $\mathbb{C}^2$.

• Draw pictures of $S_1 \cap \mathbb{R}^2$ and $S_2 \cap \mathbb{R}^2$. How accurate are these?

• Write S_i $(i = 1, 2)$ as the image of a holomorphic map $f_i : \mathbb{C} \to \mathbb{C}^2$, with f_2 bijective.

(4) Show that the analytic set $\{xw - yz = 0, xz - y^2 = 0, yw - z^2 = 0\}$ has complex codimension two in $\mathbb{C}^4$ but cannot be defined by fewer than three holomorphic equations near the origin.

[An easier but less satisfactory example: Show that the codimension one subset $\{x = 0, z = 0\}$ of the analytic set $\{xy + zw = 0\}$ cannot be cut out by fewer than two holomorphic equations.]

(5) Let $f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic, $f \neq 0$. Let $S = f^{-1}(0)$. Then S has a dense open subset U which is a smooth 2-dimensional manifold, i.e. a surface.

• Show that the restriction of the $(1,1)$-form $\omega = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$ from $\mathbb{C}^2$ to U is equal to the area form of U with respect to Riemannian metric induced by the flat metric on $\mathbb{C}^2$.

• Show that the previous statement is not true for an arbitrary surface in $\mathbb{C}^2$.

• Use the fact that $\omega = \frac{i}{2}\partial\bar{\partial}(|z|^2 + |w|^2)$ to deduce that the ratio $\frac{1}{r^2}\mathrm{Area}(U \cap B_r(0))$ is monotone increasing in r. Here $B_r(0)$ denotes the standard ball of radius r in $\mathbb{C}^2$.

• Show that U is a minimal surface in $\mathbb{C}^2$.

Exercises for Lecture 2

(1) Let (z, w) be affine coordinates on $\mathbb{C}^2$ and $(x : y)$ homogeneous coordinates on $\mathbb{P}^1$. Show that the equation $zy = wx$ cuts out a smooth complex hypersurface H in $\mathbb{C}^2 \times \mathbb{P}^1$. Describe the fibers of the holomorphic map $H \to \mathbb{C}^2$ given by projection onto the first factor.

(2) Show that the holomorphic map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by $((x_i), (y_j)) \mapsto (x_i y_j)$ is an embedding. Show that the image is the zero locus of a homogeneous quadratic polynomial.

(3) This exercise looks at holomorphic maps between projective spaces and tori.

• Show that a holomorphic map from $\mathbb{P}^n$ to a torus $\mathbb{C}^m/\Lambda$ is constant.

• Show that there is a 1-to-1 correspondence between holomorphic maps from a Riemann surface to $\mathbb{P}^1$ and meromorphic functions on the Riemann surface.

• Given a lattice $\Lambda \subset \mathbb{C}$, show that the series

$$\frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

converges and defines a nonconstant meromorphic function on $\mathbb{C}/\Lambda$.

(4) Study the Legendre family $E_t : w^2 = z(z+1)(z+t)$ of elliptic curves in $\mathbb{C}^2$.

• Draw pictures of $E_t \cap \mathbb{R}^2$. For which values of t is E_t not smooth?

• Can you visualize what happens to the Riemann surface E_t as t approaches these values?

Exercises for Lecture 3

(1) Use the presentation of $\mathcal{O}_{\mathbb{P}^n}(-1)$ in terms of charts and transition functions given in class to verify directly that $\mathcal{O}_{\mathbb{P}^n}(-1)$ has no nonzero global holomorphic sections.

(2) The tangent bundle of any Riemann surface can be given the structure of a holomorphic line bundle in a natural way. As such, $T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$. Find k.

(3) Let L be a holomorphic line bundle on some compact complex manifold.

• Show that if both L and L^* have a nonzero global holomorphic section, then both L and L^* are isomorphic to the trivial line bundle as holomorphic line bundles.

• Denoting the zero section of a line bundle by 0, prove that $L \setminus 0$ and $L^* \setminus 0$ are isomorphic as holomorphic fiber bundles with fiber $\mathbb{C}^*$, but not as holomorphic principal $\mathbb{C}^*$-bundles.

• Show that L and L^* are always isomorphic as smooth real rank two vector bundles, but not as smooth complex rank one vector bundles.

(4) Courtesy of `http://en.wikipedia.org/wiki/Theta_function`. Let τ be a complex number in the upper half plane. Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Show that the Jacobi theta function $J(z, \tau)$ can be viewed as a holomorphic section of a holomorphic line bundle L on $\mathbb{C}/\Lambda$. (From Section 9 of the Wikipedia page, L is isomorphic to $\mathcal{O}_{\mathbb{C}/\Lambda}(\frac{1+\tau}{2})$, so that $J(z, \tau)$ is the unique section of L.)

Exercises for Lecture 4

(1) Let D be a compact complex curve in a compact complex surface M. Let h be a Hermitian metric of positive curvature on $\mathcal{O}_M(D)$. Then every compact complex curve in M intersects D.

(2) Let f and g be homogeneous polynomials of degree m and n on $\mathbb{C}^3$. Consider the associated complex algebraic curves $F = \{f = 0\}$ and $G = \{g = 0\}$ in $\mathbb{P}^2$. Use integration by parts to show that if $\#(F \cap G) < \infty$, then $mn = \sum_{p \in F \cap G} \gamma_p$, where γ_p is a suitably defined "residue" at p. Then compute γ_p in some examples, preferably ones where p is a bad singularity of both F and G.

(3) This is a continuation of Exercise 4 on the previous sheet. The total space of the line bundle L of which Jacobi's theta function $w = J(z, \tau)$ is a section can be written as $(\mathbb{C} \times \mathbb{C})/\sim$, where we identify $(z, w) \sim (z + a + b\tau, e^{-\pi i b^2 \tau - 2\pi i b z} w)$ for all $a, b \in \mathbb{Z}$. Use this fact to construct a Hermitian metric h on L such that $\frac{1}{2\pi}\omega_h = \frac{i}{2}\frac{dz \wedge d\bar{z}}{\operatorname{Im}(\tau)}$, the preferred generator of $H^2(\mathbb{C}/\Lambda, \mathbb{Z})$.

(4) This exercise is meant as an introduction to Grauert's contraction theorem. We will construct some examples of rigid curves with outward convex neighborhoods on surfaces.

• Let S be the affine surface $z^2 = xy$ in $\mathbb{C}^3$, which is singular precisely at the origin. Blow up the origin in $\mathbb{C}^3$ by passing to the submanifold $H \subset \mathbb{C}^3 \times \mathbb{P}^2$ cut out by $xv = yu$ and $yw = zv$ in affine coordinates (x, y, z) on $\mathbb{C}^3$ and homogeneous coordinates $(u : v : w)$ on $\mathbb{P}^2$. Show that if $p : H \to \mathbb{C}^3$ denotes the projection, then $p^{-1}(S) = p^{-1}(0) \cup \bar{S}$, where $\bar{S}$ is a smooth surface containing a rigid curve isomorphic to $\mathbb{P}^1$: the intersection of $\bar{S}$ with $p^{-1}(0) = \mathbb{P}^2$.

• Repeat with the surface $z^3 = xy$. What changes?

Exercises for Lecture 5

(1) Let L be a holomorphic line bundle over a compact complex manifold M. Let V denote the space of global holomorphic sections of L. Show that the following hold:

• If L is basepoint free, then $\varphi_L^* \mathcal{O}(1) = L$.

• If L is basepoint free, then the divisors $s^{-1}(0)$ for $s \in V$ are precisely the preimages under φ_L of the intersections of $\varphi_L(M)$ with linear hyperplanes in $\mathbb{P}(V^*)$.

• If L is very ample, then L^k is very ample for every $k \in \mathbb{N}$.

(2) Let C be the image of $\mathbb{C}/\Lambda$ under the embedding into $\mathbb{P}^2$ associated with $\mathcal{O}(3p)$. Prove that $p_1, p_2, p_3 \in C$ lie on a straight line if and only if $\mathcal{O}(p_1) \otimes \mathcal{O}(p_2) \otimes \mathcal{O}(p_3) \cong \mathcal{O}(3p)$. One can show that this is equivalent to $p_1 + p_2 + p_3 = 3p$ in $\mathbb{C}/\Lambda$. Which points $q \in C$ satisfy $\mathcal{O}(3q) = \mathcal{O}(3p)$?

(3) For the line bundle $L = \mathcal{O}_{\mathbb{C}/\Lambda}(\frac{1+\tau}{2})$ from the previous two problem sheets, how might we try to go about constructing a section of L^* if we didn't already know that this is impossible?

(4) Show that if $n > 1$, $U \subset \mathbb{C}^n$ is open, $K \subset U$ is compact, and $U \setminus K$ is connected, then every holomorphic function on $U \setminus K$ extends to a holomorphic function on U. Some hints:

• Extend the function smoothly from $U \setminus K_\varepsilon$ to U, where K_ε denotes the closed ε-tube around K. Then the failure of this extension to be holomorphic is supported in K_ε.

• Use a cohomology vanishing theorem "with estimates": If $\alpha \in C_0^\infty(\wedge^{0,1}\mathbb{C}^n)$ and if $\bar\partial\alpha = 0$, then $\alpha = \bar\partial\beta$ for some $\beta \in C_0^\infty(\mathbb{C}^n)$ such that $\mathrm{supp}(\beta)$ is contained in the union of $\mathrm{supp}(\alpha)$ with all the bounded connected components of the complement of $\mathrm{supp}(\alpha)$.

[Another good exercise is to think about why we cannot require that $\mathrm{supp}(\beta) \subset \mathrm{supp}(\alpha)$.]

• Philosophically, this vanishing theorem holds because the trivial line bundle $\mathbb{C}^n \times \mathbb{C}$ is positive. For an actual proof, define $\beta(z_1, ..., z_n) = \frac{1}{2\pi i}\int \frac{1}{w-z_1}\alpha_1(w, z_2, ..., z_n)dw$, where $\alpha = \sum \alpha_k d\bar z^k$. Then $\bar\partial\beta = \alpha$, and $\beta = 0$ away from $\mathbb{C} \times B$ for some sufficiently large ball $B \subset \mathbb{C}^{n-1}$.

Exercises for Lecture 6

(1) Let L be a holomorphic line bundle over some complex manifold. Let h be a Hermitian metric on L. Fix $p \in M$ and a holomorphic chart $(z_1, ..., z_n)$ based at p such that $\omega_h|_p = \frac{i}{2}\sum \lambda_k dz^k \wedge d\bar z^k$ with $\lambda_k \in \mathbb{R}$. Show that there exists a local holomorphic trivialization $L \cong U \times \mathbb{C}_w$ near p such that h can be written as $e^{-\varphi}|w|^2$ with $\varphi(z) = \frac{1}{2}\sum \lambda_k |z_k|^2 + O(|z|^3)$.

(2) Let D denote the unit disk in $\mathbb{C}$ with its standard coordinate z. Let $u : D \to \mathbb{C}$ be smooth. Show that $\bar\partial^*(ud\bar z) = -u_z + \varphi_z u$ with respect to the Hermitian metric given by $h = e^{-\varphi}|w|^2$ on the trivial line bundle over D (with fiber coordinate w) and the usual flat metric on D.

(3) Some linear algebra of $(1,1)$-forms on $\mathbb{C}^n$.

• With notation as in class, verify the formulas $g(u, v) = \omega(u, Jv)$ and $g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar z^k}) = G_{jk}$.

• If ω is positive and if $W \subset \mathbb{C}^n$ is a complex subspace of complex dimension k, then show that the restriction of the $2k$-form $\frac{\omega^k}{k!}$ to W is the volume form on W induced by the metric g.

• If ω is positive with associated inner product g, and if β is another $(1,1)$-form with associated symmetric bilinear form b, then $2n\omega^{n-1} \wedge \beta = (\mathrm{trace}_g b)\omega^n$.

(4) Let G denote the subgroup of $\mathrm{GL}(2, \mathbb{C})$ generated by $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Show that the compact complex surface $M = (\mathbb{C}^2 \setminus 0)/G$ is not a Kähler manifold. Hint: Compute $H^2(M, \mathbb{R})$.

Exercises for Lecture 7

(1) Let $(L, h) \to M$ be a Hermitian holomorphic line bundle. Denote the dimension of the space of global holomorphic sections of L by $N + 1$. The precise definition of the pullback of "the Fubini-Study metric on $\mathbb{P}^N$" under φ_L is $\varphi_L^*\omega_{\mathrm{FS}} \equiv -i\partial\bar\partial \log(|s|^2/\sum |s_i|^2)$, where s is any nonvanishing local section of L, and $\{s_i\}$ is a chosen basis of global sections of L. Compare $\varphi_L^*\omega_{\mathrm{FS}}$ and ω_h.

(2) Show that the Hopf surface $(\mathbb{C}^2 \setminus 0)/\langle\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\rangle$ satisfies the global $\partial\bar\partial$-lemma.

(3) Check by computation that the Fubini-Study metric $\omega_{\mathrm{FS}} = i\partial\bar\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2)$ on $\mathbb{C}^n \subset \mathbb{P}^n$ is a Kähler-Einstein metric, i.e. that $\mathrm{Ric}(g_{\mathrm{FS}}) = \lambda g_{\mathrm{FS}}$ for some $\lambda \in \mathbb{R}$.

(4) Compute the linearization of the Monge-Ampère equation $(\omega + i\partial\bar\partial u)^n = e^f\omega^n$ at $u = 0$. It is helpful to try this on $\mathbb{C}^n$ first, where the equation can be rewritten as $\det(\delta_{jk} + \frac{\partial^2 u}{\partial z^j \partial \bar z^k}) = e^f$.

Exercises for Lecture 8

(1) Prove that $K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ by checking that if F is a homogeneous polynomial of degree $n+1$ on $\mathbb{C}^{n+1}$, then the meromorphic sections of $K_{\mathbb{P}^n}|_{U_i}$ defined by

$$(-1)^i \frac{d\frac{z_0}{z_i} \wedge \ldots \wedge d\frac{z_{i-1}}{z_i} \wedge d\frac{z_{i+1}}{z_i} \wedge \ldots \wedge d\frac{z_n}{z_i}}{z_i^{-(n+1)} F(z_0, \ldots, z_n)}$$

patch up as a global meromorphic section of $K_{\mathbb{P}^n}$ with simple poles along $F = 0$.

(2) Show that the total space of the canonical bundle of a complex manifold has trivial canonical bundle, by constructing a nowhere vanishing top degree holomorphic form on the total space.

(3) Show that a torus invariant function $\varphi = \varphi(|z|, |w|)$ on $\mathbb{C}^2$ satisfies $\det(\varphi_{j\bar{k}}) = 1$ if and only if the function $u(x, y) = \varphi(e^x, e^y)$ on $\mathbb{R}^2$ satisfies $u_{xx}u_{yy} - u_{xy}^2 = 16e^{2x+2y}$.

(4) Consider the smooth complex 3-manifold $H \subset \mathbb{C}^{2\times 2} \times \mathbb{P}^1$ cut out by the system of equations $\left(\begin{smallmatrix} x & z \\ w & y \end{smallmatrix}\right)\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. How does the map $H \to \mathbb{C}$ given by $z - w$ relate to the discussion in class?

(5) Prove that a smooth algebraic curve of degree d in $\mathbb{P}^2$ has genus $\frac{1}{2}(d-1)(d-2)$ (in the sense of surface topology) by deforming the curve to a union of d straight lines in general position.

Exercises for Lecture 9/10

(1) Compute the *moment map* of LeBrun's space $(\mathbb{C}^2, \omega)$, $\omega = i\partial\bar{\partial}\varphi$, $\varphi = u^2 + v^2 + m(u^4 + v^4)$, with respect to the holomorphic isometric circle action $\zeta(z_1, z_2) = (\zeta z_1, \bar{\zeta} z_2)$. (By definition, this is the unique function $\mu : \mathbb{C}^2 \to \mathbb{R}$ satisfying $\mu(0) = 0$ and $d\mu = X \llcorner \omega$ for the infinitesimal generator X of the circle action. Hint: $X \llcorner \omega = X \llcorner dd^c\varphi = L_X(d^c\varphi) - d(X \llcorner d^c\varphi)$.)

(2) Use integration by parts to show that if u_1 and u_2 satisfy $(\omega + dd^c u_i)^n = e^f \omega^n$ $(i = 1, 2)$ on a compact Kähler manifold, then $u_1 - u_2$ is a constant.

(3) Demystify the integration-by-parts trick for the complex Monge-Ampère equation by using a cofactor expansion to show that if $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then $\det \operatorname{Hess} u$ can be written as the divergence of a vector field depending on first and second derivatives of u.

(4) Let M^n be a compact Riemannian manifold. Let $E \to M$ be a Hermitian vector bundle with a Hermitian connection. Let A be a smooth section of the bundle $\operatorname{End} E$.

- Use Moser iteration to show that if some smooth section σ of E satisfies the Schrödinger type equation $\nabla^*\nabla\sigma + A\sigma = 0$, then $\|\sigma\|_\infty \le C(M, E)(1 + \|A\|_\infty^{n/4})\|\sigma\|_2$.
- Fix a finite-dimensional subspace V of the space of solutions to $\nabla^*\nabla\sigma + A\sigma = 0$. Pick an L^2-orthonormal basis $\{\sigma_i\}$ of V. Show that the function $\rho_V : M \to [0, \infty)$ given by $\rho_V(x) \equiv \sum |\sigma_i(x)|^2$ is independent of the choice of such a basis of V.
- Show that $\rho_V(x) = \sup\{|\sigma(x)|^2 : \sigma \in V, \|\sigma\|_2 = 1\}$.
- Deduce that $\dim V = \int_M \rho_V(x)\, d\mathrm{vol}(x) \le C(M, E)(1 + \|A\|_\infty^{n/2})$, independent of V.
- Study some special cases: eigenfunctions of the Laplacian, holomorphic sections of L^k, ...